PLANE AND SOLID ANALYTIC GEOMETRY

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PLANE ANALYTIC GEOMETRY

INTRODUCTION

DIRECTED LINE-SEGMENTS. PROJECTIONS

Elementary Geometry, as it is studied in the high school to-day, had attained its present development at the time when Greek culture was at its height. The first systematic treatment of the subject which has come down to us was written by Euclid about 300 B.C.

Algebra, on the other hand, was unknown to the Greeks. Its beginnings are found among the Hindus, to whom the so-called Arabic system of numerals may also be due. It came into Western Europe late, and not till the close of the middle ages was it carried to the point which is marked by any school book of to-day that treats this subject.

When scholars had once possessed themselves of these two subjects — Geometry and Algebra — the next step was quickly taken. The renowned philosopher and mathematician, René Descartes, in his Géometrie of 1637, showed how the methods of algebra could be applied to the study of geometry. He thus became the founder of Analytic Geometry.*

The "originals" and the locus problems of Elementary Geometry depend for their solution almost wholly on ingenuity. There are no general methods whereby one can be sure of solving a new problem of this class. Analytic Geometry,

*Also called Cartesian Geometry, from the Latinized form of his name, Cartesius.

on the other hand, furnishes universal methods for the treatment of such problems; moreover, these methods make possible the study of further problems not thought of by the ancients, but lying at the heart of modern mathematics and mathematical physics. Indeed, these two great subjects owe their very existence to the new geometry and the Calculus.

The question of how to make use in geometry of the negative, as well as the positive, numbers is among the first which must be answered in applying algebra to geometry. The solution of this problem will become clear in the following paragraphs.

1. Directed Line-Segments. Let an indefinite straight line, L, be given, and let two points, A and B, be marked on L. Then the portion of L which is bounded by A and B is what is called in Plane Geometry a *line-segment*, and is written as AB.

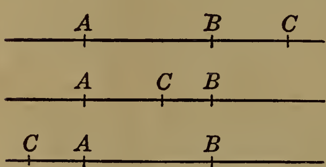

FIG. 1

Let a third point, C, be marked on L. Then three cases arise, as indicated in the figure. Corresponding to these three cases we have:

$$(a) \qquad AB + BC = AC;$$

$$(b) \qquad AB - CB = AC;$$

$$(c) \qquad CB - AB = CA.$$

Three other cases will arise if the original points A and B are taken in the opposite order on the line. Let the student write down the three corresponding equations.

A unification of all these cases can be effected by means of an extension of the concept of a line-segment. We no longer consider the line-segments AB and BA as identical, but we distinguish between them by giving each a *direction* or *sense*. Thus, AB shall be directed from A to B and BA shall be directed from B to A, *i.e.* oppositely to AB. These *directed* *line-segments* we denote by $\overline{AB}$ and $\overline{BA}$, to distinguish them from the ordinary, or undirected, line-segments.

We may, for the moment, interpret the directed line-segment $\overline{AB}$ as the act of walking from A to B; then $\overline{BA}$ represents the act of walking from B to A. With this in mind, let us return to Fig. 1 and consider the directed line-segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$. We have, in all three cases represented by Fig. 1, and also in the other three:

$$\overline{AB} + \overline{BC} = \overline{AC},$$

since walking from A to B and then walking from B to C is equivalent, with reference to the point reached, to walking from A to C.

Accordingly, we unify all six cases by defining, as the sum of the directed line-segments $\overline{AB}$ and $\overline{BC}$, the directed line-segment $\overline{AC}$:

$$(1) \qquad \overline{AB} + \overline{BC} = \overline{AC}.$$

From this definition it follows that, if A, B, C, and D are any four points of L,

$$(2) \qquad \overline{AB} + \overline{BC} + \overline{CD} = \overline{AD}.$$

For, by (1), the sum of the first two terms in (2) is $\overline{AC}$, and, by the definition, the sum of $\overline{AC}$ and $\overline{CD}$ is $\overline{AD}$.

Similarly, if the points M, M_1, M_2, $\cdots$, M_{n-1}, N are any points of L, we have

$$(3) \qquad \overline{MM_1} + \overline{M_1M_2} + \cdots + \overline{M_{n-2}M_{n-1}} + \overline{M_{n-1}N} = \overline{MN}.$$

Given two directed line-segments on the same line or on two parallel lines, we say that these two directed line-segments are *equal*, if they have equal lengths and the same direction or sense.

2. Algebraic Representation of Directed Line-Segments. On the line L let one of the two opposite directions or senses be chosen arbitrarily and defined as the *positive direction* or *sense* of L; and let the other be called the *negative direction* or *sense*.

A directed line-segment $\overline{AB}$, which lies on L, is then called *positive*, if its sense is the same as the positive sense of L, and *negative*, if its sense is the same as the negative sense of L.

To such a directed line-segment $\overline{AB}$ we assign a number, which we shall also represent by $\overline{AB}$, as follows. If l is the length of the ordinary line-segment AB, then

$$\overline{AB} = l, \qquad \text{if } \overline{AB} \text{ is a positive line-segment;}$$

$$\overline{AB} = -l, \qquad \text{if } \overline{AB} \text{ is a negative line-segment.}$$

If $\overline{AB} = l$, then $\overline{BA} = -l$; and if $\overline{AB} = -l$, then $\overline{BA} = l$. In either case

$$(1) \qquad \overline{AB} + \overline{BA} = 0 \qquad \text{or} \qquad \overline{AB} = -\overline{BA}.$$

Since the act of walking from A to B is nullified by the act of walking from B to A, we might have arrived at equations (1) from consideration of the line-segments themselves, instead of by use of the numbers which represent them.

It is easy to verify the fact that equations (1), (2), and (3) of the preceding paragraph, which relate to directed line-segments, hold for the corresponding numbers. Consequently, no error or confusion arises from using the same notation $\overline{AB}$ for both the directed line-segment and the number corresponding to it. We shall, however, adopt a still simpler notation, dropping the dash altogether and writing henceforth AB to denote, not merely the directed line-segment or the number corresponding to it, but also the line-segment itself, stating explicitly what is meant, unless the meaning is clear from the context.

Absolute Value. It is often convenient to be able to express merely the *length* of a directed line-segment, AB. The notation for this length is $|AB|$; read: "the absolute value of AB."

The numerical, or absolute, value of a number, a, is denoted in the same way: $|a|$. Thus, $|-3| = 3$. Of course, $|3| = 3$.

3. Projection of a Broken Line. By the *projection* of a point P on a line L is meant the foot, M, of the perpendicular dropped from P on L. If P lies on L, it is its own projection on L.

Let PQ be any directed line-segment, and let L be an arbitrary line. Let M and N be respectively the projections of P and Q on L. The *projection* of the directed line-segment PQ on L shall be defined as the directed line-segment MN, or the number which represents MN algebraically. Since $MN = -NM$, it follows that

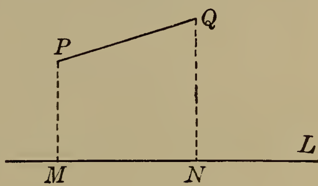

FIG. 2

$$\text{Proj. } PQ = -\text{Proj. } QP.$$

If PQ lies on a line perpendicular to L, the points M and N coincide, and we say that the projection MN of PQ on L is zero. Such a directed line-segment MN, whose end-points are identical, we may call a *nil-segment*; to it corresponds the number zero. It is evident that in taking the sum of a number of directed line-segments, any of them which are nil-segments may be disregarded, just as, in taking the sum of a set of numbers, any of them which are zero may be disregarded.

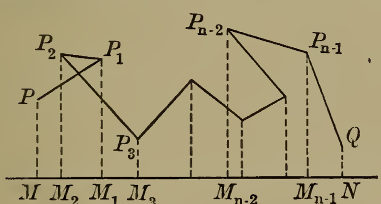

FIG. 3

Consider an arbitrary broken line $PP_1P_2 \cdots P_{n-1}Q$. By its *projection* on L is meant the sum of the projections of the directed line-segments PP_1, P_1P_2, $\cdots$, $P_{n-1}Q$, or

$$MM_1 + M_1M_2 + \cdots + M_{n-1}N.$$

This sum has the same value as MN, the projection on L of the directed line-segment PQ; cf. § 1, (3):

$$MM_1 + M_1M_2 + \cdots + M_{n-1}N = MN.$$

Hence the theorem:

THEOREM 1. *The sum of the projections on L of the segments PP_1, P_1P_2, $\cdots$, $P_{n-1}Q$ of a broken line joining P with Q is equal to the projection on L of the directed line-segment PQ.*

If, secondly, the same points P and Q be joined by another broken line, $PP'_1P'_2 \cdots P'_{m-1}Q$, the projection of the latter on L will also be equal to MN:

$$MM'_1 + M'_1M'_2 + \cdots + M'_{m-1}N = MN.$$

Hence the theorem:

THEOREM 2. *Given two broken lines having the same extremities,*

$$PP_1P_2 \cdots P_{n-1}Q \quad \textit{and} \quad PP'_1P'_2 \cdots P'_{m-1}Q.$$

Let L be an arbitrary straight line. Then the sum of the projections on L of the segments PP_1, P_1P_2, $\cdots$, $P_{n-1}Q$, of which the first broken line is made up, is equal to the corresponding sum for the second broken line.

CHAPTER I

COÖRDINATES. CURVES AND EQUATIONS

1. Definition of Rectangular Coördinates. Let a plane be given, in which it is desired to consider points and curves. Through a point O in this plane take two indefinite straight lines at right angles to each other, and choose on each line a positive sense.

Let P be any point of the plane. Consider the directed line-segment OP. Let its projections on the two directed lines through O be OM and ON. The numbers which represent algebraically these projections, that is, the lengths of OM and ON taken with the proper signs (cf. Introduction, § 2), are called the *coördinates* of P. We shall denote them by x and y:

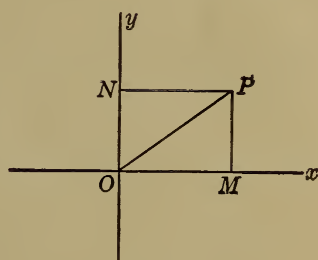

FIG. 1

$$x = OM, \qquad y = ON,$$

and write them in parentheses: (x, y). The first number, x, is known as the *x-coördinate*, or *abscissa*, of P; the second, y, as the *y-coördinate*, or *ordinate*, of P.

The point O is called the *origin* of coördinates. The directed lines through O are called the *axes of coördinates* or the *coördinate axes*; the one, the *axis of x*; the other, the *axis of y*. It is customary to take the coördinate axes as in Fig. 1, the axis of x being positive from left to right, and the axis of y, positive from below upward. But, of course, the opposite sense on one or both axes may be taken as positive, and an oblique position of the axes which conforms to the definition is legitimate, the essential thing being solely that the axes be taken *perpendicular* to each other.

Every point, P, in the plane has definite coördinates, (x, y). Conversely, to any pair of numbers, x and y, corresponds a point P whose coördinates are (x, y). This point can be constructed by laying off $OM = x$ on the axis of x, erecting a perpendicular at M to that axis, and then laying off $MP = y$. We might equally well have begun by laying off $ON = y$ on the axis of y (cf. Fig. 1), and then erected a perpendicular to that axis at N and laid off on it $NP = x$. It shall be understood that the positive sense on any line parallel to one of the coördinate axes, such as the perpendicular to the axis of x at M, shall be the same as the positive sense of that axis. For other lines of the plane there is no general principle governing the choice of the positive sense.

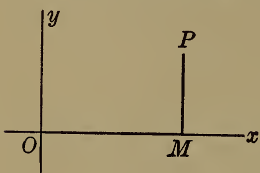

FIG. 2

The coördinates of the origin are (0, 0). Every point on the axis of x has 0 as its ordinate, and these are the only points of the plane for which this is true. *Hence the axis of x is represented by the equation*

$$y = 0, \qquad \text{(axis of } x\text{)}.$$

Similarly, the axis of y is represented by the equation

$$x = 0, \qquad \text{(axis of } y\text{)}.$$

The axes divide the plane into four regions, called *quadrants.* The *first quadrant* is the region included between the positive axis of x and the positive axis of y; the *second quadrant*, the region between the positive axis of y and the negative axis of x; etc. It is clear that the coördinates of a point in the first quadrant are both positive; that a point of the second quadrant has its abscissa negative and its ordinate positive; etc.

The system of coördinates just described is known as a *system of rectangular* or *Cartesian coördinates.*

EXERCISES

The student should provide himself with some squared paper for working these and many of the later exercises in this book. Paper ruled to centimeters and subdivided to millimeters is preferable.

1. Plot the following points, taking 1 cm. as the unit:

(*a*) $(0, 1)$; (*b*) $(1, 0)$; (*c*) $(1, 1)$;
(*d*) $(1, -1)$; (*e*) $(-1, -1)$; (*f*) $(2, -3)$;
(*g*) $(0, -2\frac{1}{2})$; (*h*) $(-3.7, 0)$; (*i*) $(-1\frac{1}{2}, -1\frac{3}{4})$;
(*j*) $(-4, 3.2)$; (*k*) $(3.24, -0.87)$; (*l*) $(-1, 1)$.

2. Determine the coördinates of the point P in Fig. 1 when 1 in. is taken as the unit of length; also when 1 cm. is the unit of length.

3. The same for the point marked by the period in "Fig. 1."

2. Projections of a Directed Line-Segment on the Axes. Let P_1, with the coördinates (x_1, y_1), and $P_2 : (x_2, y_2)$* be any two points of the plane. Consider the directed line-segment P_1P_2. It is required to find its projections on the axes.

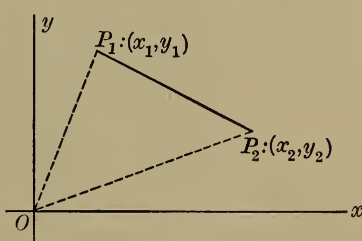

FIG. 3

To do this, draw the broken line P_1OP_2. By Introduction, § 3, Th. 1, the projections of this broken line on the axes are the same as those of the directed line-segment P_1P_2. Hence, taking first the projections on the axis of x, we have:

$$\begin{aligned}\text{Proj. } P_1P_2 &= \quad \text{Proj. } P_1O + \text{Proj. } OP_2 \\ &= -\text{Proj. } OP_1 + \text{Proj. } OP_2.\end{aligned}$$

* We shall frequently use this shorter notation, $P_2 : (x_2, y_2)$, as an abbreviation for "P_2, with the coördinates (x_2, y_2)."

But the terms in the last expression are by definition $-x_1$ and x_2. So

(1) $$\text{Proj. } P_1P_2 \text{ on } x\text{-axis} = x_2 - x_1.$$

Similarly,

(2) $$\text{Proj. } P_1P_2 \text{ on } y\text{-axis} = y_2 - y_1.$$

The projections of P_1P_2 on two lines drawn parallel to the axes are obviously given by the same expressions.

EXERCISES

1. Plot P_1P_2 when P_1 is the point (a) of Ex. 1, § 1, and P_2 is (b). Determine the projections from the foregoing formulas, and verify directly from the figure.

2. The same, when

i) P_1 is (e) and P_2 is (f);
ii) P_1 is (c) and P_2 is (d);
iii) P_1 is (i) and P_2 is (l).

3. **Distance between Two Points.** Let the points be P_1, with the coördinates (x_1, y_1), and $P_2 : (x_2, y_2)$. Through P_1 draw a line parallel to the axis of x and through P_2, a line parallel to the axis of y; let Q denote the point of intersection of these lines. Then, by the Pythagorean Theorem,

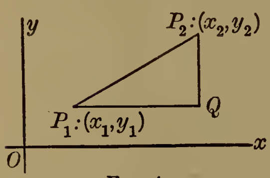

FIG. 4

(1) $$\overline{P_1P_2}^2 = \overline{P_1Q}^2 + \overline{QP_2}^2,$$

or

(2) $$D^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$

where D denotes the distance between P_1 and P_2. Hence

(3) $$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

In the foregoing analysis, we have used P_1Q (and similarly, QP_2) in two senses, namely, i) as the *length* of the ordinary line-segment P_1Q of Elementary Geometry; ii) as the algebraic expression $x_2 - x_1$ for the projection P_1Q of the directed line-segment P_1P_2 on a parallel to the axis of x. Since, however, these two numbers differ at most in sign, their *squares* are equal, and hence equation (2) is equivalent to equation (1).

In particular, P_1P_2 may be parallel to an axis, *e.g.* the axis of x. Here, $y_2 = y_1$, and (3) becomes

$$D = \sqrt{(x_2 - x_1)^2}.$$

The student must not, however, hastily infer that

$$D = x_2 - x_1.$$

It may be that $x_2 - x_1$ is negative, and then*

$$D = -(x_2 - x_1).$$

A single formula which covers both cases can be written in terms of the *absolute value* (cf. Introduction, § 2) as follows:

(4) $$D = |x_2 - x_1|.$$

EXERCISES

1. Find the distances between the following pairs of points, expressing the result correct to three significant figures. Draw a figure each time, showing the points and the line connecting them, and verify the result by actual measurement.

(*a*) (2, 1) and (– 2, – 2). (*b*) (– 7, 6) and (2, – 3).
(*c*) (13, 5) and (– 2, 5). (*d*) (7, 3) and (12, 3).
(*e*) (4, 8) and (4, – 8). (*f*) (– 1, 2) and (– 1, 6).

2. Find the lengths of the sides of the triangle whose vertices are the points (– 2, 3), (– 2, – 1), (4, – 1).

3. How far are the vertices of the triangle in question 2 from the origin?

* There is no contradiction here, or conflict with the ordinary laws of algebra. For, the $\sqrt{\ }$-sign always calls for the *positive* square root, — that being the definition of the symbol, — and we must see to it in any given case that we fulfill the contract.

4. Find the lengths of the diagonals of the convex quadrilateral whose vertices are the points (4, 1), (1, 3), (−3, 1), (−2, −1).

4. Slope of a Line. By the *slope*, λ, of a line is meant the trigonometric tangent of the angle, θ, which the line makes with the *positive* axis of x:

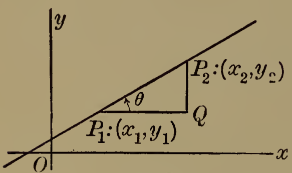

FIG. 5

(1) $$\lambda = \tan\theta.$$

To find the slope of the line, let P_1, with the coördinates (x_1, y_1), and $P_2:(x_2, y_2)$ be the extremities of any directed line-segment P_1P_2 on the line. Then

(2) $$\tan\theta = \frac{QP_2}{P_1Q} = \frac{y_2 - y_1}{x_2 - x_1},$$

or

(3) $$\lambda = \frac{y_2 - y_1}{x_2 - x_1}.$$

If, instead of P_1P_2, we had taken its opposite, P_2P_1, we should have obtained for λ the value $(y_1 - y_2)/(x_1 - x_2)$. But this is equal to the value of λ given by (3). Thus, λ is the same, whether the line is directed in the one sense or in the opposite sense. Hence we think of λ as the slope of the line without regard to sense.

Variation of the Slope. Consider the slopes, λ, of different lines, L, through a given point, P. When L is parallel to the axis of x, λ has the value zero. When L rotates as shown in the figure, λ becomes positive and increases steadily in value. As L approaches the vertical line L', λ becomes very large, increasing without limit.

When L passes beyond L', λ changes sign, being still numerically large. As L continues to rotate, λ increases *algebraically* through negative values. Finally, when L has again become parallel to the axis of x, λ has increased algebraically through all negative values and becomes again zero.

When L is in the position of L', θ is $90°$ and $\tan\theta = \lambda$ is undefined, that is, has no value. Hence L' *has no slope.* One often sees the expression: $\tan 90° = \infty$, and, in accordance with it, one might write here, $\lambda = \infty$. This does not mean that L' *has* a slope, which is infinite, for "infinity" is not a number. It is merely a brief and symbolic way of describing the behavior of λ for a line L, near to, *but not coincident with* L'; it says that for such a line λ is numerically very large; and further that, when the line L approaches L' as its limit, λ increases numerically without limit, — that is, increases numerically beyond any preassigned number, as 10,000,000 or 10,000,000!, and stays numerically above it.

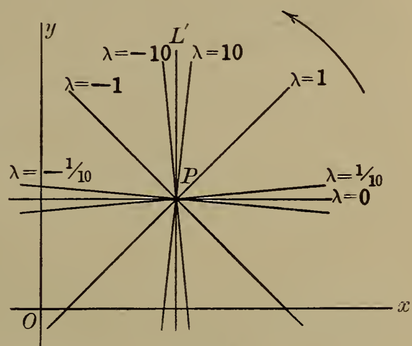

FIG. 6

The Angle θ. In measuring the angle *from* one line *to* another, it is essential, first of all, to agree on which direction of rotation shall be considered as *positive.* We shall take always as the *positive* direction of rotation that *from* the positive axis of x *to* the positive axis of y; so that the angle from the positive axis of x to the positive axis of y is $+90°$, and not $-90°$.

The complete definition of θ is, then, as follows: *The slope-angle θ of a line is the angle from the positive axis of x to the direction of the line.* There are in general two positive values for θ less than $360°$; if the smaller of them is denoted by θ, the other is $180° + \theta$. Which of these angles is chosen is immaterial, since $\tan(180° + \theta) = \tan\theta$; this result is in agreement with the previous one, to the effect that the slope pertains to the undirected line without regard to a sense on it.

The student should now draw a variety of lines, indicating for each the angle θ, and assure himself that the deduction of formula (3) holds, not merely when the quantities $x_2 - x_1$ and $y_2 - y_1$ are positive, but also when one or both are negative.

Right-Handed and Left-Handed Coördinate Systems. For the choice of axes in Fig. 1, the positive direction for angles is the counter-clockwise direction. But for such a choice as is indicated in the present figure, — a choice equally legitimate, — it is the clockwise sense which is positive.

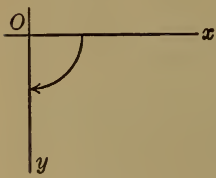

Fig. 7

The above formulas apply to either system of axes. The first system is called a *right-handed* system; the other, a *left-handed* system. We shall ordinarily use a right-handed system.

Problem. *To draw a line through a given point having a given slope.* In practice, this problem is usually to be solved on squared paper. The solution will be sufficiently clearly indicated by an example or two.

Example 1. To draw a line through the point $(-2, 3)$ having the slope -4.

Proceed along the parallel to the x-axis through the given point by any convenient distance, as 1 unit, toward the left.* Then go up the line through this point, parallel to the y-axis, by 4 times the former distance, — here, 4 units. Thus, a second point on the desired line is determined, and the line can now be drawn with a ruler.

If the given point lay near the edge of the paper, so that the above construction is inconvenient, it will do just as well to proceed from the first point toward the *right* by 1 unit, and then *down* by four units.

* The student will follow these constructions step by step on a piece of squared paper.

Example 2. To draw a line through the point (1.32, 2.78) having the slope .6541.

Here, it is clear that we cannot draw accurately enough to be able to use the last significant figure of the given slope. Open the compasses to span 10 cm. (if the squared paper is ruled to cm.) and lay off a distance of 10 cm. to the right on a parallel to the x-axis through the given point. This parallel need not actually be drawn. Its intersection, Q, with the circular arc is all that counts, and this point, Q, can be estimated and marked. Its distance above the axis of x will be 1 cm. and 3.2 mm. The error of drawing will be of the order of the last significant figure, namely, more than $\frac{1}{10}$ mm. and less than .5 mm.

Next, open the compasses to span 6 cm. and 5.4 mm. Put the point of the compasses on Q, and lay off the above distance, 6.54 cm., on a parallel through Q to the y-axis and above Q. The point R, thus found, will be a second point on the desired line, which now can be drawn.

EXERCISES

1. The points P_1, P_2, P_3, with the coördinates (2, 5), (7, 3), (−3, 7) respectively, lie on a line. Show that the value for the slope of the line as given by equation (3) is the same, no matter which two of the three points are used in obtaining it.

2. Find the slopes of the sides of the triangle of Ex. 2, § 3.

3. Find the angles which the sides of that triangle make with the axes, and hence determine the angles of the triangle.

4. Show that the points (−2, −3), (5, −4), (4, 1), (−3, 2) are the vertices of a parallelogram.

5. Draw a line through the point (1, −2) having the slope 3.

6. Draw a line through the point (−2, −1) having the slope $-1\frac{1}{2}$.

7. Draw a line through the point (−1.32, 0.14) having the slope −.2688.

5. Mid-Point of a Line-Segment. Let P_1, with the coördinates (x_1, y_1), and $P_2 : (x_2, y_2)$ be the extremities of a line-segment. It is desired to find the coördinates of the point P which bisects P_1P_2.

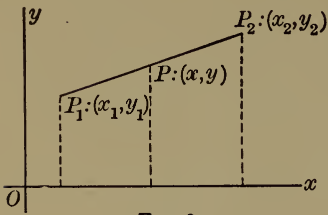

FIG. 8

Let the coördinates of P be (x, y). It is evident that the directed line-segment P_1P is equal to the directed line-segment PP_2. Hence the projection of P_1P on the axis of x, or $x - x_1$, must equal the projection of PP_2 on that axis, or $x_2 - x$:

$$x - x_1 = x_2 - x.$$

Hence

$$x = \frac{x_1 + x_2}{2}.$$

Similar considerations apply to the projections on the axis of y, and consequently

$$y = \frac{y_1 + y_2}{2}.$$

We have thus obtained the following result: *The coördinates (x, y) of the point P which bisects the line-segment P_1P_2 are given by the equations:*

(1) $$x = \frac{x_1 + x_2}{2}, \qquad y = \frac{y_1 + y_2}{2}.$$

EXERCISES

1. Determine the coördinates of the mid-point of each of the line-segments given by the pairs of points in Ex. 1, § 3. Draw figures and check your answers.

2. Find the mid-points of the sides of the triangle mentioned in Ex. 2, § 3, and check by a figure.

3. Determine the coördinates of the mid-point of the line joining the points $(a + b, a)$ and $(a - b, b)$.

4. Show that the diagonals of the parallelogram of Ex. 4, § 4 bisect each other.

6. Division of a Line-Segment in Any Ratio.* Let it be required to find the coördinates (x, y) of the point P which divides the line-segment P_1P_2 in an arbitrary ratio, m_1/m_2: †

$$\frac{P_1P}{PP_2} = \frac{m_1}{m_2}.$$

Obviously the projections of P_1P and PP_2 on the axis of x must be in the same ratio, m_1/m_2, and hence

$$\frac{x - x_1}{x_2 - x} = \frac{m_1}{m_2}.$$

On solving this equation for x, it is found that

$$x = \frac{m_2x_1 + m_1x_2}{m_2 + m_1}.$$

Similar considerations, applied to the projections on the axis of y, lead to the corresponding formula for y, and thus the coördinates of P are shown to be the following:

(1) $$x = \frac{m_2x_1 + m_1x_2}{m_2 + m_1}, \qquad y = \frac{m_2y_1 + m_1y_2}{m_2 + m_1}.$$

If m_1 and m_2 are equal, these formulas reduce to those of § 5.

External Division. It is also possible to find a point P on the indefinite straight line through P_1 and P_2 and lying outside the line-segment P_1P_2, which makes

$$\frac{P_1P}{P_2P} = \frac{m_1}{m_2},$$

where m_1 and m_2 are any two unequal positive numbers. Here,

$$\frac{x_1 - x}{x_2 - x} = \frac{m_1}{m_2}.$$

* This paragraph may well be omitted till the results are needed in later work.

† The given numbers m_1 and m_2 may be precisely the lengths P_1P and PP_2; but in general they are merely proportional respectively to them, *i.e.* they are these lengths, each multiplied by the same positive or negative number.

On solving this equation for x and the corresponding one for y, we find, as the coördinates of the point P, the following:

(2) $$x=\frac{m_2x_1-m_1x_2}{m_2-m_1}, \qquad y=\frac{m_2y_1-m_1y_2}{m_2-m_1}.$$

The point P is here said to divide the line P_1P_2 *externally* in the ratio m_1/m_2; and, in distinction, the division in the earlier case is called *internal* division. Both formulas, (1) and (2), can be written in the form (1) if one cares to consider external division as represented by a negative ratio, m_1/m_2, where, then, one of the numbers m_1, m_2 is positive, the other, negative.

EXERCISES

1. Find the coördinates of the point on the line-segment joining $(-1, 2)$ with $(5, -4)$ which is twice as far from the first point as from the second. Draw the figure accurately and verify.

2. Find the point on the line through the points given in the preceding problem, which is outside of the line-segment bounded by them and is twice as far from the first point as from the second.

3. Find the point which divides internally the line-segment bounded by the points $(3, 8)$ and $(-6, 2)$ in the ratio $1:5$, and lies nearer the first of these points.

4. The same question for external division.

7. Curve Plotting. Equation of a Curve. Since the subject of graphs is now very generally taught in the school course in Algebra, most students will already have met some of the topics taken up on the foregoing pages, and moreover they will have plotted numerous simple curves on squared paper from given equations. Thus, in particular, they will be familiar with the fact that all the points whose coördinates satisfy a *linear equation*, i.e. an *equation of the first degree*, like

(1) $$2x-3y-1=0,$$

lie on a straight line, though they may never have seen a formal proof.

A number of points, whose coördinates satisfy equation (1), can be determined by giving to x simple values, computing the corresponding values of y from (1), and then plotting the points (x, y). Thus

if $x = 0$, $y = -\frac{1}{3}$, and the point is $(0, -\frac{1}{3})$,
if $x = 1$, $y = \frac{1}{3}$, and the point is $(1, \frac{1}{3})$;
if $x = 2$, $y = 1$, and the point is $(2, 1)$;
if $x = -1$, $y = -1$, and the point is $(-1, -1)$;
etc.

Of course, if it is known that (1) represents a straight line, — *i.e.* that all the points whose coördinates satisfy (1) lie on a straight line, — it is sufficient to determine *two* points as above, and then to draw the line through them.

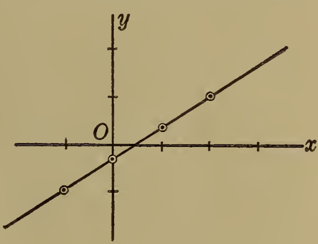

FIG. 9

This process of determining a large number of points whose coördinates satisfy a given equation and then passing a smooth curve through them is known as "plotting a curve * from its equation."

The mathematical curve † defined by an equation in x and y consists of all those points and only those points whose coördinates, when substituted for x and y in the equation, satisfy it.

Suppose, for example, that the equation is

(2) $$y = x^2.$$

The point (2, 4) lies on the curve defined by (2), because, when

* In Analytic Geometry the term *curve* includes straight lines as well as crooked curves.

† This curve is sometimes called the *locus of the equation.*

x is set equal to 2 and y is set equal to 4 in (2), the resulting equation,

$$4 = 4,$$

is true. We say, equation (2) is *satisfied* by the coördinates of the point (2, 4), or that the point (2, 4) lies on the curve (2)

On the other hand, the point (– 1, 2), for example, does not lie on the curve defined by (2). For, if we set $x = -1$ and $y = 2$, equation (2) becomes

$$2 = 1.$$

This is not a true equation; *i.e.* equation (2) is not satisfied by the coördinates of the point (– 1, 2), and so this point does not lie on the curve (2).

Equation of a Curve. A curve may be determined by simple geometric conditions; as, for example, that all of its points be at a distance of 2 units from the origin. This is a circle with its center at the origin and having a radius of length 2.

It is easy to state analytically the condition which the coördinates of any point (x, y) on the circle must satisfy. Since by § 3 the distance of any point (x, y) from the origin is

$$\sqrt{x^2 + y^2},$$

the condition that (x, y) be a point of the curve is clearly this, that

$$\sqrt{x^2 + y^2} = 2,$$

or that

(3) $$x^2 + y^2 = 4.$$

Equation (3) is called the *equation of the curve* in question.

The equation of a curve is an equation in x and y which is satisfied by the coördinates of every point of the curve, and by the coördinates of no other point.

In this book we shall be engaged for the most part in finding the equations which represent the simpler and more important curves, and in discovering and proving, from these equations, properties of the curves.

Nevertheless, the student should at the outset have clearly in mind the fact that any equation between x and y, like

$$y = x^3, \qquad y = \log x, \qquad y = \sin x,$$

represents a perfectly definite mathematical curve, which he can plot on paper. Moreover, he is in a position to determine whether, in the case of a chosen one of these curves, a given point lies on it. He will find it desirable to plot afresh a few simple curves, and to test his understanding of other matters taken up in this paragraph by answering the questions in the following exercises.

EXERCISES

1. What does each of the following equations represent? Draw a graph in each case.

(a) $x = 2$; (c) $x - y = 0$; (e) $2x - 3y + 6 = 0$;

(b) $2y + 3 = 0$; (d) $2x + 5y = 0$; (f) $5x + 8y - 4 = 0$.

Plot the following curves on squared paper.

2. $$y = x^2.$$

Take 2 cm. or 1 in. as the unit of length. Use a table of squares.

3. $$y^2 = x.$$

Take the same unit as in question 2 and use a table of square roots.

4. Show that, when one of the curves of Exs. 2 and 3 has been plotted from the tables, the other can be plotted from the first without the tables.

Work the corresponding exercises for the following curves.

5. $y = x^3$. 6. $y = \sqrt[3]{x}$. 7. $y^2 = x^3$. 8. $y^3 = x^2$.

9. Plot the curve

$$y = \log_{10} x$$

from a table of logarithms for values of x from 1 to 10, taking 1 cm. as the unit.

10. Which of the straight lines of Ex. 1 go through the origin?

11. Show that the curve

(a) $$y = \sin x$$

goes through the origin.

Do the curves

$$(b)\ y = \tan x, \qquad (c)\ y = \cos x,$$

go through the origin?

12. Do the following points lie on the curve

$$xy = 1?$$

(a) $(-1, -1)$; (b) $(-1, 1)$; (c) $(\frac{2}{3}, \frac{3}{2})$;
(d) $(-\frac{2}{3}, -\frac{3}{2})$; (e) $(\frac{1}{2}, -2)$; (f) $(0, 1)$.

13. Find the equations of the following curves.

(a) The line parallel to the axis of x and 8 units above it.

(b) The line parallel to the axis of y and $1\frac{2}{3}$ units to the left of it.

(c) The line bisecting the angle between the positive axis of y and the negative axis of x.

(d) The circle, center in the origin, radius ρ.

(e) The circle, center in the point (1, 2), radius 3.

Ans. $(x-1)^2 + (y-2)^2 = 9$.

8. Points of Intersection of Two Curves. Consider, for example, the problem of finding the point of intersection of the lines

$$L: \qquad 2x - 3y = 4,$$

$$L': \qquad 3x + 4y = -11.$$

Let (x_1, y_1) be the coördinates of this unknown point, P_1. Any point P, with the coördinates (x, y), which lies on L, has its x and y satisfying the *first* of the above equations. Hence, in particular, since P_1 lies on L, x_1 and y_1 must satisfy that equation, or

(1) $$2x_1 - 3y_1 = 4.$$

Similarly, a point $P:(x, y)$, which lies on L', has its x and y satisfying the *second* of the above equations. Hence, in particular, since P_1 lies on L', x_1 and y_1 must satisfy that equation, or

(2) $$3x_1 + 4y_1 = -11.$$

Thus it appears that the two *unknown quantities*, x_1 and y_1, satisfy the two *simultaneous equations*, (1) and (2). Hence these equations are to be solved as simultaneous by the methods of Algebra.

$$\begin{array}{rl|l} 2x_1 - 3y_1 = & 4, & 4 \\ 3x_1 + 4y_1 = & -11, & 3 \end{array}$$

To do this, eliminate y_1 by multiplying the first equation through by 4, the second by 3, and then adding:

$$17x_1 = -17, \qquad \text{or} \qquad x_1 = -1.$$

On substituting this value of x_1 in either equation (1) or (2), the value of y_1 is found to be: $y_1 = -2$. Hence P_1 has the coördinates $(-1, -2)$.

The equations (1) and (2) are the same, except for the subscripts, as the equations of the given lines, L and L'. Hence we may say: *To find the coördinates of the point of intersection of two lines given by their equations, solve the latter as simultaneous equations in the unknown quantities, x and y, by the methods of Elementary Algebra.*

The generalization to the case of any two curves given by their equations is obvious. The equations are to be regarded as *simultaneous equations between the unknown quantities, x and y*, and solved as such.

The student should observe that the letters "x" and "y" have totally different meanings when they appear as the *coordinates of a variable point in the equation of a curve*, and when they represent *unknown quantities in a pair of simultaneous equations*. In the first case, they are *variables*, and a pair of values, (x, y), which satisfy equation L will *not*, in general, satisfy L'. In the second case, x and y are *constants*, the coördinates of a *single point*, or of several points; but of *isolated* and *not variable* points.

EXERCISES

Determine the points of intersection of the following curves. Check your results by plotting the curves and reading off as accurately as possible the coördinates of the points of intersection.

1. The straight lines (*a*) and (*d*) of Ex. 1, § 7.
2. The straight lines (*c*) and (*e*) of Ex. 1, § 7.
3. The straight lines (*e*) and (*f*) of Ex. 1, § 7.

4. $\begin{cases} y^2 = 4x, \\ x + y = 3. \end{cases}$ *Ans.* (1, 2), (9, − 6).

5. $\begin{cases} x^2 + y^2 = 13, \\ xy = 6. \end{cases}$

6. $\begin{cases} x^2 + y^2 = a^2, \\ x + y = 0. \end{cases}$

7. $\begin{cases} x^2 + y^2 = 25, \\ 4x^2 + 36y^2 = 144. \end{cases}$

8. $\begin{cases} y^2 + 6x = 0, \\ 2x + y = 7. \end{cases}$

9. $\begin{cases} x^2 + y^2 = 2, \\ xy = 1. \end{cases}$ *Ans.* (1, 1), (− 1, − 1).

10. Show that the curves

$$y = \log_{10} x, \qquad x + y = 1,$$

intersect in the point (1, 0).

11. Show that the curves

$$x^2 + y^2 = 25, \qquad 3x - 4y = 0,$$

intersect in the point (4, 3), and also in (− 4, − 3).

EXERCISES ON CHAPTER I

1. Show that the points (2, 0), (0, 2), $(1 + \sqrt{3}, 1 + \sqrt{3})$ are the vertices of an equilateral triangle.

2. Prove that the triangle with vertices in the points (1, 8), (3, 2), (9, 4) is an isosceles right triangle.

3. Show that the points $(-1, 2)$, $(4, 10)$, $(2, 3)$, and $(-3, -5)$ are the vertices of a parallelogram.

4. Given the points A, B, C with coördinates $(-7, -2)$, $(-\frac{11}{5}, 0)$, $(5, 3)$. By proving that

$$AB + BC = AC,$$

show that the three points lie on a line.

5. Show that the three points of the previous problem lie on a line by proving that AB and AC have the same slope.

6. Prove that the two points $(5, 3)$ and $(-10, -6)$ lie on a line with the origin.

7. Prove that the two points (x_1, y_1), (x_2, y_2) lie on a line with the origin when, and only when, their coördinates are proportional:

$$x_1 : y_1 = x_2 : y_2.$$

8. Determine the point on the axis of x which is equidistant from the two points $(3, 4)$, $(-2, 6)$.

9. If $(3, 2)$ and $(-3, 2)$ are two vertices of an equilateral triangle which contains within it the origin, what are the coördinates of the third vertex?

10. If $(3, -1)$, $(-4, -3)$, $(1, 5)$ are three vertices of a parallelogram and the fourth lies in the first quadrant, find the coördinates of the fourth. *Ans.* $(8, 7)$.

11. If P is the mid-point of the segment P_1P_2, and P and P_1 have coördinates $(8, 17)$, $(-5, -3)$ respectively, what are the coördinates of P_2?

12. If P divides the segment P_1P_2 in the ratio $2:1$, and P_1 and P have coördinates $(3, 8)$ and $(1, 12)$ respectively, determine the coördinates of P_2. *Ans.* $(0, 14)$.

13. Find the ratio in which the point B of Ex. 4 divides the segment AC of that exercise. *Ans.* $2:3$.

14. A point with the abscissa 6 lies on the line joining the two points $(2, 5)$, $(8, 2)$. Find its ordinate.

Suggestion. Determine the ratio in which the point divides the line-segment between the two given points.

15. Prove that the sum of the squares of the distances of any point in the plane of a given rectangle to two opposite vertices equals the sum of the squares of the distances from it to the two other vertices.

Suggestion. Choose the axes of coördinates skillfully.

16. If D is the mid-point of the side BC of a triangle ABC, prove that

$$AB^2 + AC^2 = 2\,AD^2 + 2\,BD^2.$$

17. Show that the lines joining the mid-points of opposite sides of a quadrilateral bisect each other.

18. Prove that the lines joining the mid-points of adjacent sides of a quadrilateral form a parallelogram.

19. Prove that, if the diagonals of a parallelogram are equal, the parallelogram is a rectangle.

20. If two medians of a triangle are equal, show that the triangle is isosceles.

CHAPTER II

THE STRAIGHT LINE

1. Equation of Line through Two Points. Let $P_1:(x_1, y_1)$ and $P_2:(x_2, y_2)$ be two given points, and let it be required to find the equation of the line through them.

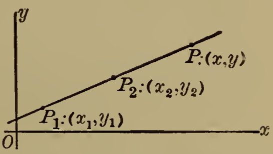

FIG. 1

The slope of the line, by Ch. I, § 4, is

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

Let P, with the coördinates (x, y), be any point on the line other than P_1. Then the slope of the line is also given by

$$\frac{y - y_1}{x - x_1}.$$

Hence

(1) $$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Conversely, if $P:(x, y)$ is any point whose coördinates satisfy equation (1), this equation then says that the slope of the line P_1P is the same as the slope of the line P_1P_2 and hence that P lies on the line P_1P_2.

A more desirable form of equation (1) is obtained by multiplying each side by $(x - x_1)/(y_2 - y_1)$. We then have:

(I) $$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}.$$

Equation (I) is satisfied by the coördinates of those points and only those points which lie on the line P_1P_2. Consequently by Ch. 1, § 7, (I) *is the equation of the line through the two given points.*

Example 1. Find the equation of the line which passes through the points (1, – 2) and (– 3, 4).

Here

$$x_1 = 1, \quad y_1 = -2 \qquad \text{and} \qquad x_2 = -3, \quad y_2 = 4.$$

By (I) the equation of the line is

$$\frac{x-1}{-3-1} = \frac{y-(-2)}{4-(-2)}, \qquad \text{or} \qquad \frac{x-1}{-4} = \frac{y+2}{6}.$$

On clearing of fractions and reducing, the equation becomes

$$3x + 2y + 1 = 0.$$

Let the student show that, if (x_1, y_1) had been taken as (– 3, 4) and (x_2, y_2) as (1, – 2), the same equation would have resulted.

Example 2. Find the equation of the line passing through the origin and the point (a, b).

Here, $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (a, b)$, and (I) becomes

$$\frac{x}{a} = \frac{y}{b}, \qquad \text{or} \qquad bx - ay = 0.$$

Lines Parallel to the Axes. In deducing (I) we tacitly assumed that

$$y_2 - y_1 \neq 0 \qquad \text{and} \qquad x_2 - x_1 \neq 0;$$

for otherwise we could not have divided by these quantities.

If $y_2 - y_1 = 0$, the line is parallel to the axis of x. Its equation is, then, obviously

(2) $$y = y_1.$$

Similarly, if $x_2 - x_1 = 0$, the line is parallel to the axis of y and has the equation

(3) $$x = x_1.$$

These two special cases are not included in the result embodied in equation (I). We see, however, that they are so simple, that they can be dealt with directly.*

Example 3. Find the equation of the line passing through the two points $(-5, 1)$ and $(-5, 8)$.

It is clear from the figure that this line is parallel to the axis of y and 5 units distant from it to the left. Accordingly, the abscissa of every point on it is -5; conversely, every point whose abscissa is -5 lies on it. Therefore, its equation is

$$x = -5, \qquad \text{or} \qquad x + 5 = 0.$$

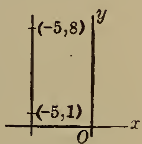

FIG. 2

EXERCISES †

Draw the following lines and find their equations.

1. Through (1, 1) and (3, 4). *Ans.* $3x - 2y - 1 = 0$.
2. Through (5, 3) and $(-8, 6)$.
3. Through $(0, -5)$ and $(-2, 0)$. *Ans.* $5x + 2y + 10 = 0$.
4. Through the origin and $(-1, 2)$.
5. Through the origin and $(-2, -3)$.
6. Through $(2, -3)$ and $(-4, -3)$. *Ans.* $y + 3 = 0$.

* It is not difficult to replace (I) by an equation which holds in *all* cases, — namely, the following:

(I′) $$(y_2 - y_1)(x - x_1) = (x_2 - x_1)(y - y_1).$$

We prefer, however, the original form (I). For (I) is more compact and easier to remember, and the special cases not included in it are best handled without a formula.

† In substituting numerical values for (x_1, y_1) and (x_2, y_2) in (I), the student will do well to begin with a framework of the form

$$\frac{x -}{-} = \frac{y -}{-},$$

and then fill in each place in which x_1 occurs; next, each place in which y_1 occurs; and so on. When x_1 or y_1 is negative, substitute it first in parentheses; thus, if $x_1 = -3$, begin by writing

$$\frac{x-(-3)}{-(-3)} = \frac{y -}{-}.$$

7. Through (0, 8) and (0, − 56).

8. Through (5, 3) and parallel to the axis of y.

9. Through (5, 3) and parallel to the axis of x.

10. Through (a, b) and (b, a). *Ans.* $x + y = a + b$

11. Through $(a, 0)$ and $(0, b)$. *Ans.* $\frac{x}{a} + \frac{y}{b} = 1$.

2. One Point and the Slope Given. Let it be required to find the equation of the line which passes through a given point $P_1 : (x_1, y_1)$ and has a given slope, λ.

If $P : (x, y)$ be any second point on the line, the slope of the line will be, by Ch. I, § 4,

$$\frac{y - y_1}{x - x_1}.$$

But the slope of the line is given as λ. Hence

$$\frac{y - y_1}{x - x_1} = \lambda,$$

or

(II) $$y - y_1 = \lambda(x - x_1).$$

The student can now show, conversely, that any point, whose coördinates (x, y) satisfy (II), lies on the given line. Hence (II) is the equation of the line passing through the given point and having the given slope.

Example. Find the equation of the line which goes through the point (2, − 3) and makes an angle of 135° with the positive axis of x.

Here, $\lambda = -1$ and $(x_1, y_1) = (2, -3)$, and hence, by (II), the equation of the line is

$$y + 3 = -1(x - 2),$$

or

$$x + y + 1 = 0.$$

Slope-Intercept Form of Equation. It is frequently convenient to determine a line by its slope λ, and the y-coördinate of the point in which it cuts the axis of y.

Here, $x_1 = 0$; and, if we denote y_1 by the letter b, (II) becomes

(III) $$y = \lambda x + b.$$

This is known as the *slope-intercept form* of the equation of a straight line; b is known as the *intercept* of the line on the axis of y.

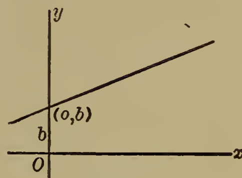

FIG. 3

Example. Find the equation of the line which makes an angle of 60° with the axis of x and whose intercept on the axis of y is -2.

Since $\lambda = \sqrt{3}$ and $b = -2$, the equation is

$$y = \sqrt{3}\,x - 2.$$

EXERCISES

Draw the following lines and find their equations.

1. Through $(-4, 5)$ and with slope -2.

Ans. $2x + y + 3 = 0$.

2. Through $(3, 0)$ and with slope $\frac{8}{3}$.

3. Through $(\frac{2}{5}, -\frac{1}{2})$ and with slope $-\frac{5}{3}$.

4. Through the origin and making an angle of 60° with the axis of x.

5. Through $(-4, 0)$ and making an angle of 45° with the axis of y.

6. With intercept 1 on the axis of y and with slope $-\frac{3}{2}$.

Ans. $3x + 2y - 2 = 0$.

7. With intercept $\frac{1}{2}$ on the axis of y and making an angle of 30° with the axis of x.

8. With slope -1 and intercept $-c$ on the axis of y.

9. With slope a/b and intercept b on the axis of y.

Ans. $ax - by + b^2 = 0$.

3. The General Equation of the First Degree. Let there be given an arbitrary line of the plane. If the line is parallel to neither axis, its equation is of the form (I), § 1, — an equation of the first degree in x and y. If the line is parallel to the axis of x, its equation is of the form $y = y_1$, — a special equation of the first degree in x and y, in which it happens that the term in x is lacking. Similarly, if the line is parallel to the axis of y, its equation is of the form $x = x_1$, — an equation of the first degree which lacks the term in y. Consequently, we can say: *The equation of every straight line is of the first degree in x and y.*

Given, conversely, *the general equation of the first degree* in x and y, namely

(1) $$Ax + By + C = 0,$$

where A, B, C are any three constants, of which A and B are not both zero;* *this equation represents always a straight line.*

The Case $B \neq 0$. In general, B will not be zero and we can divide equation (1) through by it:

$$\frac{A}{B}x + y + \frac{C}{B} = 0,$$

and then solve for y:

$$y = -\frac{A}{B}x - \frac{C}{B}.$$

But this equation is precisely of the form (III), § 2, where

$$\lambda = -\frac{A}{B}, \qquad b = -\frac{C}{B}.$$

Therefore, it represents a straight line whose slope is $-A/B$ and whose intercept on the axis of y is $-C/B$.

The Case $B = 0$. If, however, B is zero, the equation (1) becomes

$$Ax + C = 0.$$

Now, A cannot be zero, since the case that both A and B are zero was excluded at the outset. We can, therefore, divide by A and then solve for x:

$$x = -\frac{C}{A}.$$

* In dealing with equation (1), now and henceforth, we shall always assume that A and B are not both zero.

This is the equation of a straight line parallel to the axis of y, if $C \neq 0$. If $C = 0$, it is the equation of this axis.

This completes the proof that every equation of the first degree represents a straight line. In accordance with this property, such an equation is frequently called a *linear* equation.

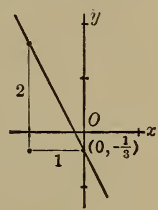

FIG. 4

Example. What line is represented by the equation

$$6x + 3y + 1 = 0?$$

If we solve for y, we obtain

$$y = -2x - \tfrac{1}{3}.$$

Hence the equation represents the line of slope -2 with intercept $-\frac{1}{3}$ on the axis of y. From these data we may draw the line.

EXERCISES

Find the slopes and the intercepts on the axis of y of the lines represented by the following equations. Draw the lines.

1. $4x + 2y - 1 = 0$.
2. $7x + 8y + 5 = 0$.
3. $2x - 5y = 0$.
4. $2x - y = 1$.
5. $y = 0$.
6. $x = 3 - y$.

Find the slopes of each of the following lines.

7. $-x + 2y = 7$. *Ans.* $\frac{1}{2}$.
8. $x = y + 1$.
9. $3 - 2x = 5y$.
10. $2x - 3y = 4$.
11. $2y - 3 = 0$.
12. $2x = 3y$.
13. $x = 5y + 1$.
14. $bx + ay = ab$.

4. Intercepts. In the preceding paragraph we learned to plot the line represented by a given equation, from the values of its slope and its intercept on the axis of y, as found from the equation. It is often simpler, however, in the case of a line which cuts the axes in two distinct points, to determine from the equation the coördinates of these two points and then to plot the points and draw the line through them.

The point of intersection of a line, for example,

$$(1) \qquad 2x - 3y + 4 = 0,$$

with the axis of x has its y-coördinate equal to 0. Consequently, to find the x-coördinate of the point, we have but to set $y = 0$ in the equation of the line and solve for x. In this case we have, then,

$$2x + 4 = 0, \qquad \text{or} \qquad x = -2.$$

Similarly, the x-coördinate of the point of intersection of the line with the axis of y is 0, and its y-coördinate is obtained by setting $x = 0$ in the equation of the line and solving for y. In the present case this gives

$$-3y + 4 = 0, \qquad \text{or} \qquad y = \tfrac{4}{3}.$$

The points of intersection of the line (1) with the axes of coördinates are, then, $(-2, 0)$ and $(0, \frac{4}{3})$. We now plot these points and draw the line through them.

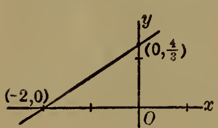

FIG. 5

We recognize the number $\frac{4}{3}$ as the intercept of the line (1) on the axis of y; the number -2 we call the intercept on the axis of x. We have plotted the line (1), then, by finding its intercepts.

In general, the *intercept* of a line on the axis of x is the x-coördinate of the point in which the line meets that axis. The intercept on the axis of y is similarly defined. These definitions admit of extension to any curve. Thus, the circle of Ch. I, § 7, has two intercepts on the axis of x, namely, $+2$ and -2.

An axis or a line parallel to an axis has no intercept on that axis. Every other line has definite intercepts on both axes, and these intercepts determine the position of the line unless they are both zero, that is, unless the line goes through the origin.

EXERCISES

Determine the intercepts of the following lines on each of the coördinate axes, so far as such intercepts exist, and draw the lines.

1. $2x + 3y - 6 = 0$. *Ans.* 3, 2.
2. $x - y + 1 = 0$.
3. $3x - 5y + 10 = 0$.
4. $5x + 7y + 13 = 0$.
5. $2x - 3y = 0$.
6. $2x + 3 = 0$. *Ans.* $-1\frac{1}{2}$, none.
7. $8 - 5y = 0$.
8. $x = 0$.
9. $x + y = a$.
10. $2ax - 3by = ab$.

5. The Intercept Form of the Equation of a Line. Given a line whose position is determined by its intercepts. Let the intercept on the axis of x be a, and let that on the axis of y be b. To find the equation of the line in terms of a and b.

Since one point on this line is $(a, 0)$ and a second is $(0, b)$, we have, by (I), § 1,

$$\frac{x - a}{0 - a} = \frac{y - 0}{b - 0},$$

or

(IV) $$\frac{x}{a} + \frac{y}{b} = 1.$$

Only lines which intersect the axes in two points that are distinct can have their equations written in this form. A line through the origin is an exception, because one or both its intercepts are zero and division by zero is impossible. Also a line parallel to an axis is an exception, since it has no intercept on that axis.

EXERCISES

Find the equations of the following lines.

1. With intercepts 5 and 3.
2. With intercepts $-2\frac{1}{2}$ and 8.
3. With intercepts $\frac{4}{5}$ and $-\frac{3}{4}$.

4. The diagonals of a square lie along the coördinate axes, and their length is 2 units. Find the equations of the four sides (produced).

Ans. $x + y = 1$; $x - y = 1$; $-x + y = 1$; $-x - y = 1$.

5. A triangle has its vertices at the points $(0, 1)$, $(-2, 0)$, $(1, 0)$. Draw the triangle and find the equations of its sides (produced). Use formula (IV), when possible.

6. A triangle has its vertices at the points $(a, 0)$, $(b, 0)$, $(0, c)$. Find the equations of the sides (produced).

7. A line goes through the origin and the mid-point of that side of the triangle of Ex. 5 which lies in the first quadrant. Find its equation.

8. Find the equations of the lines through the origin and the respective mid-points of the sides of the triangle of Ex. 6.

6. Parallel and Perpendicular Lines. *Parallels.* Given two lines oblique to the axis of y, so that both have slopes. The lines are parallel if, and only if, they have equal slopes. For, if they are parallel, their slope angles, and hence their slopes, are equal; and conversely.

Example 1. To find the equation of the line through the point $(1, 2)$ parallel to the line

(1) $$3x - 2y + 6 = 0.$$

The slope of the line (1) is $\frac{3}{2}$. The required line has the same slope and passes through the point $(1, 2)$. By (II), § 2, its equation is

$$y - 2 = \tfrac{3}{2}(x - 1),$$

or

$$3x - 2y + 1 = 0.$$

If the given line is parallel to the axis of y, it has no slope and hence the method of Example 1 is inapplicable. But then the required line must also be parallel to the axis of y and its equation can be written down directly. For example, if the given line is $3x + 8 = 0$, and there is required the line parallel to it passing through the point $(-8, 2)$, it is clear that the required line is parallel to the axis of y and 8 units to the left of it, and consequently has the equation $x = -8$, or $x + 8 = 0$.

Perpendiculars. Given two lines oblique to the axes, so that both have slopes, neither of which is zero. The lines are perpendicular if, and only if, their slopes, λ_1 and λ_2, are *negative reciprocals* of one another:

$$(2) \qquad \lambda_2 = -\frac{1}{\lambda_1}, \qquad \text{or} \qquad \lambda_1 = -\frac{1}{\lambda_2}, \qquad \lambda_1 \neq 0,\ \lambda_2 \neq 0.$$

For, if the lines are perpendicular, one of their slope angles, θ_1 and θ_2, may be taken as 90° greater than the other, viz.:

$$\theta_2 = \theta_1 + 90°,$$

and hence

$$\lambda_2 = \tan\theta_2 = \tan(\theta_1 + 90°) = -\cot\theta_1 = -\frac{1}{\tan\theta_1} = -\frac{1}{\lambda_1},$$

or

$$\lambda_2 = -\frac{1}{\lambda_1}.$$

Conversely, if this last equation is valid, the steps can be retraced and the lines shown to be perpendicular to each other.

Example 2. To find the equation of the line through the point (1, 2) perpendicular to the line (1).

The slope of (1) is $\frac{3}{2}$. Hence the required line has the slope $-\frac{2}{3}$. We have, then, to find the equation of the line through the point (1, 2) with slope $-\frac{2}{3}$. By (II), § 2, this equation is

$$y - 2 = -\tfrac{2}{3}(x - 1),$$

or

$$2x + 3y - 8 = 0.$$

If the given line is parallel to an axis, it has no slope or its slope is zero. In either case, equation (2) and the method of Example 2 are inapplicable. But then the required line must be parallel to the other axis and it is easy to write its equation. Suppose, for example, that the given line is $2y - 3 = 0$, — a line parallel to the axis of x, — and that the required line perpendicular to it is to go through the point (3, 5). Then this line must be parallel to the axis of y and at a distance of 3 units to the right of it. Consequently, its equation is $x - 3 = 0$.

The methods of this paragraph are applicable to all problems in which it is required to find the equation of a line which passes through a given point and is parallel, or perpendicular, to a given line.

EXERCISES

In each of the following exercises find the equations of the lines through the given point parallel and perpendicular to the given line.

	Line	*Point*
1.	$4x - 8y = 5,$	$(-1, -3).$ *Ans.* $x - 2y - 5 = 0;\quad 2x + y + 5 = 0.$
2.	$x - y = 1,$	$(0, 0).$
3.	$5x + 13y - 3 = 0,$	$(2, -1).$
4.	$3x + 5y = 0,$	$(5, 0).$
5.	$2x = 3,$	$(5, -6).$
6.	$\sqrt{2}y + \pi = 0,$	$(-2, 0).$ *Ans.* $y = 0;\quad x + 2 = 0.$
7.	$1 - x = 0,$	$(0, \pi).$

8. Find the equations of the altitudes of the triangle of § 5, Ex. 5.

9. Find the equations of the perpendicular bisectors of the sides of the triangle of § 5, Ex. 5.

10. Show that the equation of the line through the point (x_1, y_1) parallel to the line

$$Ax + By = C \tag{3}$$

is

$$Ax + By = Ax_1 + By_1.$$

11. Show that the equation of the line through the point (x_1, y_1) perpendicular to the line (3) of Ex. 10 is

$$Bx - Ay = Bx_1 - Ay_1.$$

7. Angle between Two Lines. Let L_1 and L_2 be two given lines, whose slopes are, respectively,

$$\lambda_1 = \tan\theta_1, \qquad \text{and} \qquad \lambda_2 = \tan\theta_2.$$

To find the angle, ϕ, *from* L_1 *to* L_2.

Since

$$\phi = \theta_2 - \theta_1,$$

it follows from Trigonometry that

$$\tan \phi = \frac{\tan \theta_2 - \tan \theta_1}{1 + \tan \theta_1 \tan \theta_2},$$

and hence that

(1) $$\tan \phi = \frac{\lambda_2 - \lambda_1}{1 + \lambda_1 \lambda_2}.$$

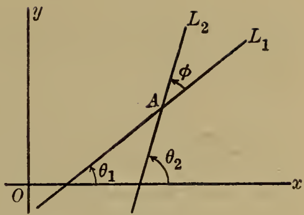

FIG. 6

The angle ϕ is the angle *from* L_1 *to* L_2. That is, it is the angle through which L_1 must be rotated *in the positive sense*, about the point A, in order that it coincide with L_2. In particular, we agree to take it as the smallest such angle, always less, then, than 180°: $0 \leq \phi < 180°$.*

If L_1 and L_2 are perpendicular, then, by (2), § 6, $\lambda_2 = -1/\lambda_1$ and $1 + \lambda_1\lambda_2 = 0$. Consequently, cot ϕ, which is equal to the reciprocal of the right-hand side of (1), has the value zero, and so $\phi = 90°$.

Example. Let L_1 and L_2 be given by the equations,

L_1: $$4x - 2y + 7 = 0,$$

L_2: $$12x + 4y - 5 = 0.$$

Here $\lambda_1 = 2$ and $\lambda_2 = -3$, and (1) becomes

$$\tan \phi = \frac{-3-2}{1-6} = 1.$$

Hence the angle ϕ from L_1 to L_2 is 45°.

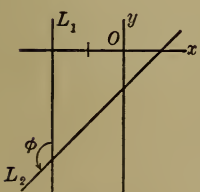

FIG. 7

In deducing (1) it was assumed that L_1 and L_2 both have slopes. If this is not the case, at least one of the lines is parallel to the axis of y and no formula is needed. The angle ϕ may be found directly. Suppose, for example, that L_1 and L_2 are, respectively,

$$x + 2 = 0 \qquad \text{and} \qquad x - y = 1.$$

* The figure shows L_1 and L_2 as intersecting lines, but formula (1) and the deduction of it are valid also in case L_1 and L_2 are parallel. In this

Then L_1 is parallel to the axis of y, and L_2 is inclined at an angle of 45° to the positive axis of x, since $\lambda_2 = 1$. Consequently, $\phi = 135°$.

EXERCISES

In each of the following exercises determine whether the given lines are mutually parallel or perpendicular, and in case they are neither, find the angle from the first line to the second.

1. $x + 2y = 3,$ $\quad x + 2y = 4.$
2. $2x - y + 5 = 0,$ $\quad 4x - 2y - 7 = 0.$
3. $x - y = 1,$ $\quad x + y = 2.$
4. $x + 2y + 11 = 0,$ $\quad 6x - 3y - 4 = 0.$
5. $3x - y = 0,$ $\quad 2x + y = 0.$
6. $x + 2y + 1 = 0,$ $\quad 2x + y - 1 = 0.$
7. $4x + 3y = 3,$ $\quad 9x - 3y = 5.$
8. $2x - 3y = 1,$ $\quad x - 3 = 0.$
9. $x + y = 0,$ $\quad y = 0.$
10. $2x - 3y + 1 = 0,$ $\quad 3x - 4y - 1 = 0.$

11. By the method of this paragraph determine each of the three angles of the triangle whose sides have the equations

$$x - 2y - 6 = 0, \qquad 2x + y - 4 = 0, \qquad 3x - y + 3 = 0.$$

Check your results by adding the angles.

12. Prove that if L_1 and L_2 are represented by the equations

$$L_1: \qquad A_1x + B_1y + C_1 = 0,$$

$$L_2: \qquad A_2x + B_2y + C_2 = 0,$$

then

$$\tan\phi = \frac{A_1B_2 - A_2B_1}{A_1A_2 + B_1B_2}.$$

What can you say of L_1 and L_2 if $A_1B_2 - A_2B_1 = 0$? If $A_1A_2 + B_1B_2 = 0$?

case, we take the angle from L_1 to L_2 as 0° —, not as 180°, as is conceivable. Hence arises the sign $\leq$ (less than or equal to) in the place in which it stands in the double inequality.

13. Show that the formula of Ex. 12 for $\tan\phi$ is valid even if one or both of the lines has no slope, *i.e.* is parallel to the axis of y.

8. Distance of a Point from a Line. Let $P:(x_1, y_1)$ be a given point and let

$$L: \qquad Ax + By + C = 0$$

be a given line. To find the distance, D, of P from L.

Drop a perpendicular from P on the axis of x, and denote the point in which it cuts L by Q. The abscissa of Q is x_1. Denote its ordinate by y_Q. Then

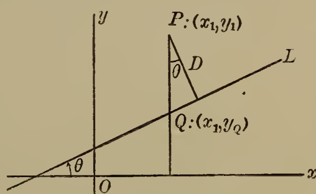

FIG. 8

$$QP = y_1 - y_Q.$$

Since $Q:(x_1, y_Q)$ lies on L, its coördinates satisfy the equation of L; thus

$$Ax_1 + By_Q + C = 0.$$

Solving this equation for y_Q, we find:

$$y_Q = -\frac{Ax_1 + C}{B}.$$

Hence

$$(1) \qquad QP = \frac{Ax_1 + By_1 + C}{B}.$$

Let θ be the slope-angle of L and form the product $QP\cos\theta$. One or both of the factors of this product may be negative, according to the positions of P and L.* But always the numerical value of the product is equal to the distance D:

$$(2) \qquad D = |QP\cos\theta|.$$

This is clear in case P and L are situated as in Fig. 8;

* There are four essentially different positions for P and L, for L may have a positive or a negative slope, and P may lie on the one or on the other side of L.

the student should draw the other typical figures and show that for them, also, (2) is valid.

Since the slope of L is

$$\lambda = \tan\theta = -\frac{A}{B},$$

we have

$$\sec^2\theta = 1 + \tan^2\theta = \frac{A^2 + B^2}{B^2}.$$

Consequently,

$$\cos\theta = \pm\frac{B}{\sqrt{A^2 + B^2}}. \tag{3}$$

It is immaterial to us which sign in (3) is the proper one. For, according to (2), we have now to multiply together the values of QP and $\cos\theta$, as given by (1) and (3), and take the *numerical* value of the product. The result is the desired formula:

$$\text{(4)}\quad \begin{cases} D = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}, \\ \text{or} \\ D = \pm\dfrac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}, \end{cases}$$

where, in the second formula, that sign is to be chosen which makes the right-hand side positive.

Example. The distance of the point $(3, -2)$ from the line

$$3x + 4y - 7 = 0$$

is

$$D = \frac{|3\cdot 3 + 4(-2) - 7|}{\sqrt{3^2 + 4^2}} = \frac{|-6|}{\sqrt{25}} = \frac{6}{5} = 1\tfrac{1}{5}.$$

The deduction of formula (4) involves division by B and hence tacitly assumes that $B \neq 0$, *i.e.* that L is not parallel to the axis of y. The formula holds, however, even when L is parallel to the axis of y. For, in this case it is clear from a figure that

$$D = \left|x_1 + \frac{C}{A}\right|,$$

and (4) reduces precisely to this when $B = 0$.

EXERCISES

In each of the first seven exercises find the distance of the given point from the given line.

	Point	*Line*	
1.	(5, 2),	$3x - 4y + 6 = 0.$	*Ans.* $2\frac{3}{5}$.
2.	(2, 3),	$5x + 12y + 2 = 0.$	
3.	(6, − 1),	$3x - y + 1 = 0.$	*Ans.* $2\sqrt{10}$, or 6.32.
4.	(3, 4),	$3x + 5 = 0.$	
5.	(− 2, − 5),	$y = 0.$	
6.	Origin,	$x + y - 1 = 0.$	
7.	Origin,	$3x + 2y - 6 = 0.$	

8. Find the lengths of the altitudes of the triangle with vertices in the points (2, 0), (3, 5), (− 1, 2).

9. Area of a Triangle. Let a triangle be given by means of its vertices (x_1, y_1), (x_2, y_2), (x_3, y_3). To find its area.

Drop a perpendicular from one of the vertices, as (x_3, y_3), on the opposite side. Then the required area is

$$A = \tfrac{1}{2} DE,$$

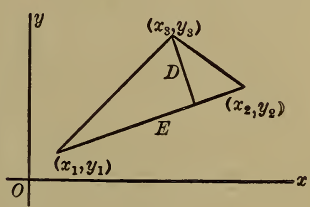

FIG. 9

where D denotes the length of the perpendicular and E, the length of the side in question.

By Ch. I, § 3, we have

$$E = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

D is the distance of (x_3, y_3) from the line joining (x_1, y_1) and (x_2, y_2). The equation of this line, as given by (I) or (I′), § 1, may be put into the form:

$$(y_2 - y_1)x - (x_2 - x_1)y - x_1y_2 + x_2y_1 = 0.$$

Consequently, by (4), § 8, we find:

$$D = \pm \frac{(y_2 - y_1)x_3 - (x_2 - x_1)y_3 - x_1y_2 + x_2y_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}.$$

Thus

$$A = \pm \tfrac{1}{2}[(y_2 - y_1)x_3 - (x_2 - x_1)y_3 - x_1y_2 + x_2y_1].$$

The result may be written more symmetrically in either of the forms

(1) $$A = \pm \tfrac{1}{2}[(x_1 - x_2)y_3 + (x_2 - x_3)y_1 + (x_3 - x_1)y_2],$$

or

(2) $$A = \pm \tfrac{1}{2}[(y_1 - y_2)x_3 + (y_2 - y_3)x_1 + (y_3 - y_1)x_2],$$

where in each case that sign is to be chosen which makes the right-hand side positive.

EXERCISES

Find the area of the triangle whose vertices are in the points

1. $(1, 2)$, $(-1, 2)$, $(-2, 1)$.
2. $(5, 3)$, $(-3, 4)$, $(-2, -1)$.
3. $(1, 2)$, $(2, 1)$, $(0, 0)$.

Find the area of the triangle whose sides lie along the lines

4. $x - y = 0$, $\quad x + y = 0$, $\quad 2x + y - 3 = 0$.
5. $2x + y - 6 = 0$, $\quad x - y + 3 = 0$, $\quad x - 2y - 8 = 0$.
6. Find the area of the convex quadrilateral whose vertices are in the points $(4, 2)$, $(-1, 4)$, $(-3, -2)$, $(5, -8)$.
7. What do formulas (1) and (2) become when one of the vertices, say (x_3, y_3), is in the origin?

Ans. $A = \pm \tfrac{1}{2}(x_1y_2 - x_2y_1)$.

10. General Theory of Parallels and Perpendiculars. Identical Lines.* The line through the point (x_1, y_1) parallel to the line

(1) $$Ax + By = C,$$

has the equation, according to § 6, Ex. 10,

$$Ax + By = Ax_1 + By_1.$$

* The discussion in the class-room of the subjects treated in this and the following paragraph may well be postponed until the need for them arises.

This equation is of the form

(2) $$Ax + By = C',$$

since the constant $Ax_1 + By_1$ may be denoted by the single letter C'.

Conversely, equations (1) and (2), for $C' \neq C$, always represent parallel lines. For, if $B \neq 0$, the lines have the same slope, $-A/B$; if $B = 0$, A cannot be zero, and the lines are parallel to the axis of y and hence to each other.

THEOREM 1. *Two lines are parallel when and only when their equations can be written in the forms* (1) *and* (2), *where* $C \neq C'$.

The line through the point (x_1, y_1), perpendicular to the line (1), has the equation (§ 6, Ex. 11):

$$Bx - Ay = Bx_1 - Ay_1,$$

and this equation is of the form

(3) $$Bx - Ay = C'.$$

Let the student show, conversely, that equations (1) and (3) always represent perpendicular lines.

THEOREM 2. *Two lines are perpendicular when and only when their equations can be written in the forms* (1) *and* (3).

The equations of two parallel lines *can* always be written in the forms (1) and (2). But they need not be so written. Thus the lines,

$$2x - y = -1,$$
$$6x - 3y = 2,$$

are parallel, though the equations are not in the forms (1) and (2). The coefficients of the terms in x and y are not respectively equal. They are, however, proportional: $2 : 6 = -1 : -3$.

This condition holds in all cases. For the two lines

L_1: $$A_1x + B_1y + C_1 = 0,$$
L_2: $$A_2x + B_2y + C_2 = 0,$$

we may state the theorem:

THEOREM 3. *The lines L_1 and L_2 are parallel* if and only if*

$$A_1 : A_2 = B_1 : B_2.$$

For, L_1 and L_2 are parallel if and only if the angle ϕ between them, as defined in § 7, is zero; but, according to § 7, Ex. 12, ϕ, or better, $\tan\phi$, is zero, when and only when $A_1B_2 - A_2B_1 = 0$. But this equation is equivalent to the proportion $A_1 : A_2 = B_1 : B_2$.

As a second consequence of § 7, Ex. 12, we obtain the following theorem.

THEOREM 4. *The lines L_1 and L_2 are perpendicular if and only if*

$$A_1A_2 + B_1B_2 = 0.$$

Identical Lines. Two equations do not have to be identically the same in order to represent the same line. For example, the equations,

$$2x - y + 1 = 0,$$
$$6x - 3y + 3 = 0,$$

represent the same line. The corresponding constants in them are not equal, but they are proportional. We have, namely,

$$2 : 6 = -1 : -3 = 1 : 3,$$

or, what amounts to the same thing,

$$2 : -1 : 1 = 6 : -3 : 3.$$

This condition is general. We formulate it as a theorem:

THEOREM 5. *The lines L_1 and L_2 are identical if and only if*

$$A_1 : A_2 = B_1 : B_2 = C_1 : C_2,$$

or

$$A_1 : B_1 : C_1 = A_2 : B_2 : C_2.$$

For, L_1 and L_2 are the same line when and only when they have the same slope and the same intercept on the axis of y, that is, when and only when

$$-\frac{A_1}{B_1} = -\frac{A_2}{B_2} \quad \text{and} \quad -\frac{C_1}{B_1} = -\frac{C_2}{B_2},$$

* Or, in a single case, identical. Cf. Th. 5.

or $A_1 : A_2 = B_1 : B_2$ and $B_1 : B_2 = C_1 : C_2$,

or, finally, $A_1 : A_2 = B_1 : B_2 = C_1 : C_2$.

This proof assumes that $B_1 \neq 0$ and $B_2 \neq 0$. The proof, when this is not the case, is left to the student.

EXERCISES

1. Prove Th. 3 directly, without recourse to the results of § 7.

2. The same for Th. 4.

See also Exs. 15, 16, 17, 18 at the end of the chapter.

11. Second Method of Finding Parallels and Perpendiculars.

Problem 1. To find the equation of a line parallel to the given line

(1) $$Ax + By = C,$$

and satisfying a further condition.

By § 10, Th. 1, the desired equation can be written in the form

(2) $$Ax + By = C',$$

where C' is to be determined by the further condition.

Example. Consider the first example treated in § 6. In this case the equation of the desired line can be written in the form

$$3x - 2y = k,$$

where we have replaced the C' of (2) by k. The "further condition," by means of which the value of k is to be determined, is that the line go through the point (1, 2). Hence $x = 1$, $y = 2$ must satisfy the equation of the line, or

$$3 \cdot 1 - 2 \cdot 2 = k.$$

Consequently, $k = -1$, and the equation of the line is

$$3x - 2y + 1 = 0.$$

Problem 2. To find the equation of a line perpendicular to the given line (1) and satisfying a further condition.

By § 10, Th. 2, the desired equation can be written in the form

$$(3) \qquad Bx - Ay = C',$$

where C' is to be determined by applying the further condition.

This condition does not always have to be that the line should go through a given point. It may be any single condition, not affecting the slope of the line, which it seems desirable to apply. We give an example illustrating the method in such a case.

Example. To find the equation of the line perpendicular to

$$2x - y - 4 = 0$$

and cutting from the first quadrant a triangle whose area is 16.

Equation (3) may, in this case, be written as

$$(4) \qquad x + 2y = k.$$

We are to determine k so that the line (4) cuts from the first quadrant a triangle of area 16. The intercepts of the line (4) are k and $\frac{1}{2}k$, and hence the area of the triangle in question is $\frac{1}{4}k^2$. Accordingly, $\frac{1}{4}k^2 = 16$, and $k = \pm 8$. But the line cuts the first quadrant only if k is positive, and so we must have $k = 8$. The equation of the desired line is, then,

$$x + 2y - 8 = 0.$$

EXERCISES

1. Work Exs. 1–4, 8, 9 of § 6 by this method.

2. There are two lines parallel to the line

$$x - 2y = 6$$

and forming with the coördinate axes triangles of area 9. Find their equations.

3. Find the equations of the lines parallel to the line of Ex. 2 and 3 units distant from it.

Suggestion. Write the equation of the required line in the form (2) and demand that the distance from it of a chosen point of the given line be 3.

4. Find the equations of the lines parallel to the line

$$5x + 12y - 3 = 0$$

and 2 units distant from the origin.

5. The same as Ex. 2, if the lines are to be perpendicular, instead of parallel, to the given line.

6. The same as Ex. 4, if the lines are to be perpendicular, instead of parallel, to the given line.

7. A line is parallel to the line $3x + 2y - 6 = 0$, and forms a triangle in the first quadrant with the lines,

$$x - 2y = 0 \quad \text{and} \quad 2x - y = 0,$$

whose area is 21. Find the equation of the line.

Ans. $3x + 2y - 28 = 0$.

EXERCISES ON CHAPTER II

1. Find the equation of the line whose intercepts are twice those of the line $2x - 3y - 6 = 0$.

2. Find the equation of the line having the same intercept on the axis of x as the line $\sqrt{3}x - y - 3 = 0$, but making with that axis half the angle.

3. Find the equation of the line joining the point $(3, -2)$ with that point of the line $2x - y = 8$ whose ordinate is 2.

4. A perpendicular from the origin meets a line in the point $(5, 2)$. What is the equation of the line?

5. The coördinates of the foot of the perpendicular dropped from the origin on a line are (a, b). Show that the equation of the line is

$$ax + by = a^2 + b^2.$$

6. The line through the point $(5, -3)$ perpendicular to a given line meets it in the point $(-3, 2)$. Find the equation of the given line.

7. Prove that the line with intercepts 6 and 3 is perpendicular to the line with intercepts 3 and -6. Is it also perpendicular to the line with intercepts -3 and 6?

8. Prove that the line with intercepts a and b is perpendicular to the line with intercepts b and $-a$.

9. Show that the two points (5, 2) and (6, -15) subtend a right angle at the origin.

10. Prove that the two points, (x_1, y_1) and (x_2, y_2), subtend a right angle at the origin when, and only when, $x_1x_2 + y_1y_2 = 0$.

11. Do the points (6, -1) and (-3, 4) subtend a right angle at the point (4, 6)? At the point (-4, -2)?

12. Given the triangle whose sides lie along the lines,

$$x - 2y + 6 = 0, \qquad 2x - y = 3, \qquad x + y - 3 = 0.$$

Find the coördinates of the vertices and the equations of the lines through the vertices parallel to the opposite sides.

13. Two sides of a parallelogram lie along the lines,

$$2x + 3y - 6 = 0, \qquad 4x - y = 4.$$

A vertex is at the point (-2, 1). Find the equations of the other two sides (produced).

14. One side of a rectangle lies along the line,

$$5x + 4y - 9 = 0.$$

A vertex on this side is at the point (1, 1) and a second vertex is at (2, -1). Find the equations of the other three sides (produced).

15. For what value of λ will the two lines,

$$3x - 2y + 6 = 0, \qquad \lambda x - y + 2 = 0,$$

(a) be parallel? (b) be perpendicular?

16. For what value, or values, of m will the two lines,

$$4x - my + 6 = 0, \qquad x + my + 3 = 0,$$

(a) be parallel? (b) be perpendicular?

17. For what value of m will the two equations,

$$mx + y + 5 = 0, \qquad 4x + my + 10 = 0,$$

represent the same line?

18. For what pairs of values for k and l will the two equations,

$$12x + ky + l = 0, \qquad lx - 5y + 3 = 0,$$

represent the same line?

19. The equations of the sides of a convex quadrilateral are

$$x = 2, \qquad y = 4, \qquad y = x, \qquad 2y = x.$$

Find the coördinates of the vertices and the equations of the diagonals.

20. Find the equation of the line through the point of intersection of the lines,

$$3x - 5y - 11 = 0, \qquad 2x - 7y = 11,$$

and having the intercept -5 on the axis of y.

21. Find the equation of the line through the point of intersection of the lines,

$$2x + 5y = 4, \qquad 3x - 4y + 17 = 0,$$

and perpendicular to the first of these two lines.

22. Find the distance between the two parallel lines,

$$3x - 4y + 1 = 0, \qquad 6x - 8y + 9 = 0.$$

Suggestion. Find the distance of a chosen point of the first line from the second.

23. Let

$$Ax + By + C = 0 \quad \text{and} \quad Ax + By + C' = 0$$

be any two parallel lines. Show that the distance between them is

$$\frac{|C' - C|}{\sqrt{A^2 + B^2}}, \qquad \text{or} \qquad \pm\frac{C' - C}{\sqrt{A^2 + B^2}}.$$

24. There are two points on the axis of x which are at the distance 4 from the line $2x - 3y - 4 = 0$. What are their coördinates?

25. Find the coördinates of the point on the axis of y which is equidistant from the two points $(3, 8)$, $(-2, 5)$.

26. There are two lines through the point (1, 1), each cutting from the first quadrant a triangle whose area is $2\frac{1}{4}$. Find their slopes. *Ans.* $-\frac{1}{2}, -2$.

27. Find the equation of the line through the point (3, 7) such that this point bisects the portion of the line between the axes. *Ans.* $7x + 3y - 42 = 0$.

28. The origin lies on a certain line and is the mid-point of that portion of the line intercepted between the two lines,

$$3x - 5y = 6, \qquad 4x + y + 6 = 0.$$

Find the equation of the line. *Ans.* $x + 6y = 0$.

29. The line

$$(1) \qquad 3x - 8y + 5 = 0$$

goes through the point (1, 1). Find the equation of the line (2) through this same point, if the angle from the line (1) to the line (2) is 45°. *Ans.* $11x - 5y - 6 = 0$.

30. Find the equations of the two lines through the origin making with the line $2x - 3y = 0$ angles of 60°.

CHAPTER III

APPLICATIONS

1. Certain General Methods. *Lines through a Point.* In many theorems and problems of Plane Geometry the question is to show that *three lines pass through a point.* Plane Geometry affords, however, no general method for dealing with this question. Each new problem must be discussed as if it were the first of this class to be considered.

Analytic Geometry, on the other hand, affords a universal method, whereby in any given case the question can be settled. For, from the data of the problem, the equation of each of the lines can be found. These will all be linear, and can be written in the form

$$L_1: \qquad A_1x + B_1y + C_1 = 0,$$

$$L_2: \qquad A_2x + B_2y + C_2 = 0,$$

$$L_3: \qquad A_3x + B_3y + C_3 = 0.$$

The coördinates of the point of intersection of two of these lines, as L_1 and L_2, can be found by solving the corresponding equations, regarded as simultaneous, for the unknown quantities x and y. Let the solution be written as

$$x = x', \qquad y = y'.$$

The third line, L_3, will pass through this point (x', y'), if and only if the coördinates of the latter satisfy the equation of L_3; i.e. *if and only if*

$$A_3x' + B_3y' + C_3 = 0.$$

Points on a Line. A second question which presents itself in problems of Plane Geometry is to determine when *three* *points lie on a straight line.* Here, again, the reply of Analytic Geometry is methodical and universal. From the data of the problem it will be possible in any given case to obtain the coördinates of the three points. Call them

$$(x_1, y_1), \qquad (x_2, y_2), \qquad (x_3, y_3).$$

Now, we know how to write down the equation of a line through two of them, as (x_1, y_1) and (x_2, y_2). This equation will always be linear, and can be written in the form

$$Ax + By + C = 0.$$

The third point, (x_3, y_3), will lie on this line if and only if its coördinates satisfy the equation of the line; i.e. *if and only if*

$$Ax_3 + By_3 + C = 0.$$

The student should test his understanding of the foregoing theory by working Exs. 1–6 at the end of the chapter.

2. The Medians of a Triangle. We recall the proposition from Plane Geometry, that *the medians of a triangle meet in a point.* The proof there given is simple, provided one remembers the construction lines it is desirable to draw. By means, however, of Analytic Geometry we can establish the proposition, not by artifices, but by the natural and direct application of the general principle enunciated in the preceding paragraph.

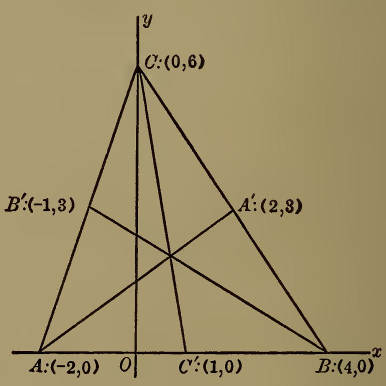

Fig. 1

The first step consists in the choice of the coördinate axes. This choice is wholly in our hands, and we make it in such a way as to simplify the coördinates of the given points. Thus, clearly, it will be well to take one of the axes along a side of the triangle. Let this be the axis of x.

A good choice for the axis of y will be one in which this axis passes through a vertex. Let this be the vertex not on the axis of x.

We begin with a numerical case, choosing the vertices A, B, C at the points indicated in the figure.

The Equations of the Medians. Consider the median AA'. One point on this line is given, namely $A:(-2, 0)$. A second point is the mid-point A' of the line-segment BC. By Ch. I, § 5, the coördinates of A' are $(2, 3)$.

The student can now solve for himself the problem of finding the equation of the line L_1 through $A:(-2, 0)$ and $A':(2, 3)$. The answer is,

$$L_1: \qquad 3x - 4y + 6 = 0.$$

In a precisely similar way the coördinates of B' are found to be $(-1, 3)$, and the equation of the median BB' is

$$L_2: \qquad 3x + 5y - 12 = 0.$$

Finally, the coördinates of C' are $(1, 0)$, and the equation of the median CC' is

$$L_3: \qquad 6x + y - 6 = 0.$$

The Point of Intersection of the Medians. The next step consists in finding the point in which two of the medians, as L_1 and L_2, intersect. The coördinates of this point will be given by solving as simultaneous the equations of these lines:

$$3x - 4y + 6 = 0,$$
$$3x + 5y - 12 = 0.$$

The solution is found to be:

$$x = \tfrac{2}{3}, \qquad y = 2.$$

And now the third median, L_3, will go through this point, $(\frac{2}{3}, 2)$, if the coördinates of the point satisfy the equation of L_3,

$$6x + y - 6 = 0.$$

On substituting for x in this equation the value $\frac{2}{3}$ and for y the value 2, we are led to the equation

$$6 \cdot \tfrac{2}{3} + 2 - 6 = 0.$$

This is a true equation, and hence the three lines L_1, L_2, and L_3 pass through the same point.

Remark. It can be shown by the formulas of Ch. I, § 6, that the above point $(\frac{2}{3}, 2)$ trisects each of the medians AA', BB', and CC'.

EXERCISES

1. Taking the same triangle as before, choose the axis of x along the side AB, but take the axis of y through A. The coördinates of the vertices will then be:

$$A:(0, 0); \qquad B:(6, 0); \qquad C:(2, 6).$$

Prove the theorem for this triangle.

2. The vertices of a triangle lie at the points (0, 0), (3, 0), (0, 9). Prove that the medians meet in a point.

3. Continuation. The General Case. We now proceed to prove the theorem of the medians for any triangle, ABC. Let

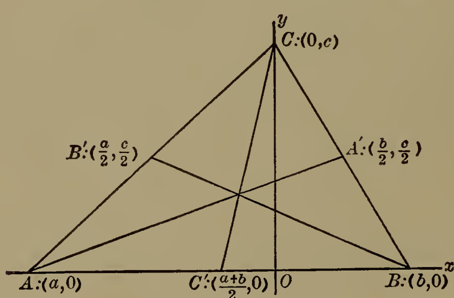

Fig. 2

the axes be chosen as in the text of § 2. Then the coördinates of A will be $(a, 0)$, where a may be any number whatever, positive, negative, or zero. The coördinates of B will be $(b, 0)$, where b may be any number distinct from a:

$$b \neq a, \qquad \text{or} \qquad a - b \neq 0.^*$$

Finally, the coördinates of C can be written as $(0, c)$, where c is any positive number.

Next, find the coördinates of A', B', C'. They are as shown in the figure.

The equation of L_1 is given by Ch. II, (I), where

$$(x_1, y_1) = (a, 0); \qquad (x_2, y_2) = \left(\frac{b}{2}, \frac{c}{2}\right).$$

It is:

$$\frac{x - a}{\frac{b}{2} - a} = \frac{y - 0}{\frac{c}{2} - 0},$$

or

L_1: $$cx + (2a - b)y = ac.$$

The equation of L_2 can be worked out in a similar manner. But it is not necessary to repeat the steps, since interchanging the letters a and b interchanges the points A and B, and also A' and B'. Thus L_1 passes over into L_2. Hence the equation of L_2 is:

L_2: $$cx + (2b - a)y = bc.$$

The line L_3 is determined by its intercepts, $\frac{1}{2}(a + b)$ and c: by Ch. II, (IV), its equation is found to be:

L_3: $$2cx + (a + b)y = (a + b)c.$$

To find the coördinates of the point in which L_1 and L_2 intersect, solve as simultaneous the equation of L_1 and L_2:

$$\begin{cases} cx + (2a - b)y = ac, \\ cx + (2b - a)y = bc. \end{cases}$$

The result is:

$$x = \frac{a + b}{3}, \qquad y = \frac{c}{3}.$$

* The figure has been drawn for the case in which a is negative and b positive.

Finally, to show that this point, $\left(\frac{a+b}{3}, \frac{c}{3}\right)$, lies on L_3, substitute its coördinates in the equation of L_3:

$$2c\frac{a+b}{3}+(a+b)\frac{c}{3}=(a+b)c.$$

Since this is a true equation, the point lies on the line, and we have proved the theorem that *the medians of a triangle pass through a point.*

That this point trisects each median can be proved as in the special case of the preceding paragraph, by means of Ch. I, § 6. The details are left to the student.

EXERCISE

Prove the theorem of the medians by taking the coördinate axes as in the first exercise of the preceding paragraph. Here, the vertices are

$$A:(0, 0); \qquad B:(a, 0); \qquad C:(b, c),$$

where a may be any number not 0, b any number whatever, and c any positive number. Draw the figure, and write in the coördinates of each point used.

4. The Altitudes of a Triangle. Another proposition of Plane Geometry is, that *the perpendiculars dropped from the vertices of a triangle on the opposite sides meet in a point.*

The proof of the proposition by Analytic Geometry is direct and simple. Let us begin with a numerical case, taking the triangle of Fig. 1. One of the perpendiculars is, then, the axis of y, and so all that is necessary to show is that the other two meet on this axis, or that the x-coördinate of their point of intersection is 0.

The equation of the line BC can be written down at once in terms of its intercepts:

$$\frac{x}{4}+\frac{y}{6}=1, \qquad \text{or} \qquad 3x+2y=12.$$

The slope of this line is $\lambda = -\frac{3}{2}$. The slope of any line perpendicular to it is $\lambda' = \frac{2}{3}$. Hence the equation of L_1, the perpendicular which passes through the point $A:(-2, 0)$, is

$$y - 0 = \tfrac{2}{3}(x+2),$$

or

$$L_1: \qquad 2x - 3y + 4 = 0.$$

In a similar manner the student can obtain the equation of the perpendicular L_2 from B on the side AC. It is,

$$L_2: \qquad x + 3y - 4 = 0.$$

On computing the x-coördinate of the point in which L_1 and L_2 intersect, it is found that $x = 0$, and hence the proposition is established for this triangle.

Remark. For use in a later problem it is necessary to know the exact point in which the perpendiculars meet. It is readily shown that this point is $(0, \frac{4}{3})$.

EXERCISES

1. Prove the above proposition for the special triangle considered, choosing the coördinate axes as in Ex. 1 of § 2.

2. Prove the proposition for the triangle of Ex. 2, § 2.

3. Prove the proposition for the general case, choosing the axes as in Fig. 2. First show that the equation of the perpendicular L_1 from A on BC is

$$L_1: \qquad bx - cy = ab,$$

and that the equation of the perpendicular L_2 from B on AC is

$$L_2: \qquad ax - cy = ab.$$

Then show that these lines intersect each other on the axis of y.

4. Show that the point in which the perpendiculars in the preceding question meet is $\left(0, -\dfrac{ab}{c}\right)$.

5. Prove the theorem of the altitudes, when the axes of coördinates are taken as in the exercise of § 3.

5. The Perpendicular Bisectors of the Sides of a Triangle. It is shown in Plane Geometry that these lines meet in a point. Since the student is now in full possession of the method employed in Analytic Geometry for the proof of this theorem, he will find it altogether possible to work out that proof without further suggestion. Let him begin with the special triangle of Fig. 1. He will find that the equations of the perpendicular bisectors of the sides are the following:

L_1: $$2x - 3y + 5 = 0;$$
L_2: $$x + 3y - 8 = 0;$$
L_3: $$x - 1 = 0.$$

These lines are then shown to meet in the point $(1, \frac{7}{3})$.

He can work further special examples corresponding to the exercises at the end of § 2 if this seems desirable.

Finally, let him work out the proof for the general case, taking the coördinate axes as in Fig. 2. The three lines will be found to have the equations

L_1: $$bx - cy = \tfrac{1}{2}(b^2 - c^2),$$
L_2: $$ax - cy = \tfrac{1}{2}(a^2 - c^2),$$
L_3: $$x = \tfrac{1}{2}(a + b).$$

They meet in the point

$$\left(\frac{a+b}{2}, \frac{ab+c^2}{2c}\right).$$

EXERCISE

Give the proof when the axes of coördinates are taken as in the exercise of § 3.

6. Three Points on a Line. The foregoing three propositions about triangles have led to three points, namely, the three points of intersection of the three lines in the various cases. In the case of the special triangle of Fig. 1, these points are

$$(\tfrac{2}{3}, 2); \qquad (0, \tfrac{4}{3}); \qquad (1, \tfrac{7}{3}).$$

These points lie on a straight line. Let the student try to prove this theorem by Plane Geometry.

The proof by Analytic Geometry is given immediately as a direct application of the second of the general principles enunciated in the opening paragraph of the chapter.

Write down the equation of the line through two of these points, — say, through the first and third. It is found to be:

$$3x - 3y + 4 = 0.$$

The coördinates of the second point,

$$x = 0, \qquad y = \tfrac{4}{3},$$

are seen to satisfy this equation, and the proposition is proved.

EXERCISES

1. Prove the proposition for the general case (Fig. 2). The points have been found to be:

$$\left(\frac{a+b}{3}, \frac{c}{3}\right); \quad \left(0, -\frac{ab}{c}\right); \quad \left(\frac{a+b}{2}, \frac{ab+c^2}{2c}\right).$$

2. On plotting the three points obtained in the special case discussed in the text it is observed that the line-segment determined by the extreme points is divided by the intermediate point in the ratio of 1 : 2. Prove this analytically. Is it true in general?

EXERCISES ON CHAPTER III

1. Prove that the three lines,

$$2x - 3y - 5 = 0, \quad 3x + 4y - 16 = 0, \quad 4x - 23y + 7 = 0,$$

go through a point.

2. Prove that the three lines,

$$ax + by = 1, \quad bx + ay = 1, \quad x - y = 0,$$

go through a point.

3. Prove that the three points (4, 1), (− 1, − 9), and (2, − 3) lie on a line.

4. Prove that the three points (a, b), (b, a), and $(-a, 2a+b)$ lie on a line.

5. Find the condition that the three lines,

$$bx + ay = 2ab, \quad ax + by = a^2 + b^2, \quad 3x - 2y = 0,$$

where a^2 is not equal to b^2, meet in a point.

6. Find the condition that the three points (a, b), (b, a), and $(2a, -b)$, where a is not equal to b, lie on a line.

Lines through a Point

7. Show that the line drawn through the mid-points of the parallel sides of a trapezoid passes through the point of intersection of the non-parallel sides.

8. Show that, in a trapezoid, the diagonals and the line drawn through the mid-points of the parallel sides meet in a point.

9. A right triangle has its vertices A, B, and O in the points $(4, 0)$, $(0, 3)$, and $(0, 0)$. The points $A' : (4, -4)$ and $B' : (-3, 3)$ are marked. Prove that the lines AB', BA', and the perpendicular from O on the hypothenuse meet in a point.

10. (Generalization of Ex. 9.) Given a right triangle ABO with the right angle at O. On the perpendicular to OA in the point A measure off the distance AA', equal to OA, in the direction away from the hypothenuse. In a similar fashion mark the point B' on the perpendicular to OB in B, so that $BB' = OB$. Prove that the lines AB', BA', and the perpendicular from O on the hypothenuse meet in a point.

11. Let P be any point (a, a) of the line $x - y = 0$, other than the origin. Through P draw two lines, of arbitrary slopes λ_1 and λ_2, intersecting the x-axis in A_1 and A_2 and the y-axis in B_1 and B_2 respectively. Prove that the lines A_1B_2 and A_2B_1 will, in general, meet on the line $x + y = 0$.

12. If on the three sides of a triangle as diagonals parallelograms, having their sides parallel to two given lines, are

described, the other diagonals of the parallelograms meet in a point.

Prove this theorem, when the given lines are the coördinate axes, and the triangle has as its vertices the points (1, 6), (4, 11), (9, 3).

13. Prove the theorem of the preceding exercise, when the given lines are the axes, and the triangle has its vertices in the points (0, 0), (a, a), (b, c).

Points on a Line

14. Show that in the parallelogram $ABCD$ the vertex D, the mid-point of the side AB, and a point of trisection of the diagonal AC lie on a line.

15. Prove that the feet of the perpendiculars from the point $(2, -1)$ on the sides of the triangle with vertices in the points (0, 0), (3, 0), and (0, 1) lie on a line.

16. Prove that the feet of the perpendiculars from the point $(-1, 4)$ on the sides of the triangle with vertices in the points (2, 0), $(-3, 0)$, and (0, 4) lie on a line.

17. Show that the feet of the perpendiculars from the point $\left(0, \dfrac{ab}{c}\right)$ on the sides of the triangle with vertices in the points $(a, 0)$, $(b, 0)$, and $(0, c)$ lie on a line.

18. Let M be the point of intersection of two opposite sides of a quadrilateral, and N, the point of intersection of the other two sides. The mid-point of MN and the mid-points of the diagonals lie on a right line.

Prove this proposition for the special case that the vertices of the quadrilateral are situated at the points (0, 0), (8, 0), (6, 4), (1, 6).

19. Prove the proposition of Ex. 18 for the general case.

Suggestion. Take the axis of x through M and N, the origin being at the mid-point. The equations of the sides can then be written in the form

$$y = \lambda_1(x - h), \qquad y = \lambda_2(x - h),$$
$$y = \lambda_3(x + h), \qquad y = \lambda_4(x + h).$$

20. Let O be the foot of the altitude from the vertex C of the triangle ABC on the side AB. Then the feet of the perpendiculars from O on the sides BC and AC and on the other two altitudes lie on a line.

Prove this theorem for the triangle ABC with vertices in the points $(1, 0)$, $(-4, 0)$, $(0, 2)$.

21. Prove the theorem of the preceding exercise for the triangle with vertices in the points $(a, 0)$, $(b, 0)$, $(0, c)$. It will be found that

$$\left(\frac{ac^2}{a^2+c^2}, \frac{a^2c}{a^2+c^2}\right), \left(\frac{bc^2}{b^2+c^2}, \frac{b^2c}{b^2+c^2}\right),$$
$$\left(\frac{a^2b}{a^2+c^2}, \frac{-abc}{a^2+c^2}\right), \left(\frac{ab^2}{b^2+c^2}, \frac{-abc}{b^2+c^2}\right),$$

are the coördinates of the four points which are to lie on a line, and that

$$c(a + b)x + (ab - c^2)y = abc$$

is the equation of the line.

CHAPTER IV

THE CIRCLE

1. Equation of the Circle. According to Ch. I, § 7, the equation of the circle whose center is at the origin, and whose radius is ρ, is

(1) $$x^2 + y^2 = \rho^2.$$

In a precisely similar manner, the equation of a circle with its center at an arbitrary point $C:(\alpha, \beta)$ of the plane, the length of the radius being denoted by ρ, is found to be:

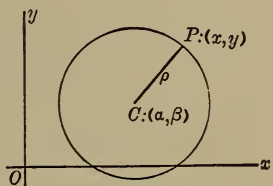

FIG. 1

(2) $$(x-\alpha)^2 + (y-\beta)^2 = \rho^2.$$

Example. Find the equation of the circle whose center is at the point $(-\frac{4}{3}, 0)$, and whose radius is $\frac{2}{3}$.

Here, $\alpha = -\frac{4}{3}$, $\beta = 0$, and $\rho = \frac{2}{3}$. Hence, from (2):

$$(x + \tfrac{4}{3})^2 + y^2 = \tfrac{4}{9}.$$

This equation can be simplified as follows:

$$x^2 + \tfrac{8}{3}x + \tfrac{16}{9} + y^2 = \tfrac{4}{9},$$

or, finally,

$$3x^2 + 3y^2 + 8x + 4 = 0.$$

EXERCISES

Find the equations of the following circles, and reduce the results to their simplest form. Draw the figure each time.

1. Center at (4, 6); radius, 3.

Ans. $x^2 + y^2 - 8x - 12y + 43 = 0.$

2. Center at (0, −2); radius, 2. *Ans.* $x^2 + y^2 + 4y = 0.$

3. Center at $(-3, 0)$; radius, 3.

4. Center at $(2, -4)$; radius, 8.

5. Center at $(0, \frac{4}{3})$; radius, $\frac{2}{3}$.

6. Center at $(3, -4)$; radius, 5.

7. Center at $(-5, 12)$; radius, 13.

8. Center at $(\frac{7}{5}, -\frac{2}{5})$; radius, 2.

9. Center at $(-\frac{5}{4}, \frac{8}{3})$; radius, $\frac{11}{6}$.

10. Center at $(a, 0)$; radius, a.

11. Center at $(0, a)$; radius, a.

12. Center at (a, a); radius, $a\sqrt{2}$.

2. A Second Form of the Equation. Equation (2) of § 1 can be expanded as follows:

$$x^2 + y^2 - 2\alpha x - 2\beta y + \alpha^2 + \beta^2 - \rho^2 = 0.$$

This equation is of the form

$$(1) \qquad x^2 + y^2 + Ax + By + C = 0.$$

Let us see whether, conversely, equation (1) always represents a circle.

Example 1. Determine the curve represented by the equation

$$(2) \qquad x^2 + y^2 + 2x - 6y + 6 = 0.$$

We can rewrite this equation as follows:

$$(x^2 + 2x \qquad) + (y^2 - 6y \qquad) = -6.$$

The first parenthesis becomes a perfect square if 1 is added; the second, if 9 is added. To keep the equation true, these numbers must be added also to the right-hand side. Thus

$$(x^2 + 2x + 1) + (y^2 - 6y + 9) = -6 + 1 + 9,$$

or

$$(x + 1)^2 + (y - 3)^2 = 4.$$

This equation is precisely of the form (2), § 1, where $\alpha = -1$, $\beta = 3$, $\rho = 2$. It therefore represents a circle whose center is at $(-1, 3)$, and whose radius is 2.

Example 2. What curve is represented by the equation

$$x^2 + y^2 + 1 = 0 ? \tag{3}$$

It is clear that *no* point exists whose coördinates satisfy this equation. For, x^2 and y^2 can never be negative. Their least values are 0, — namely, for the origin, (0, 0), — and even for this point, the left-hand side of the equation has the value + 1. *Hence, there is no curve corresponding to equation* (3).

Example 3. Discuss the equation

$$x^2 + y^2 + 2x - 4y + 5 = 0. \tag{4}$$

Evidently, this equation can be written in the form:

$$(x + 1)^2 + (y - 2)^2 = 0. \tag{5}$$

The coördinates of the point (− 1, 2) satisfy the equation. But, for any other point (x, y), at least one of the quantities, $x + 1$ and $y - 2$, is not zero, and the left-hand side of the equation is positive. Thus the point (− 1, 2) is the only point whose coördinates satisfy the equation. *Hence equation* (4) *represents a single point* (− 1, 2).

Remark. Equation (5) can be regarded as the limiting case of the equation

$$(x + 1)^2 + (y - 2)^2 = \rho^2,$$

when ρ approaches the limit 0. This equation represents a circle of radius ρ for all positive values of ρ. When ρ approaches 0, the circle shrinks down toward the point (− 1, 2) as its limit. Accordingly, equation (5) is sometimes spoken of as representing a circle of zero radius or a *null circle.*

The General Case. It is now clear how to proceed in the general case, in order to determine what curve equation (1) represents. The equation can be written in the form:

$$(x^2 + Ax + \tfrac{1}{4}A^2) + (y^2 + By + \tfrac{1}{4}B^2) = - C + \tfrac{1}{4}A^2 + \tfrac{1}{4}B^2,$$

or

$$\left(x + \frac{A}{2}\right)^2 + \left(y + \frac{B}{2}\right)^2 = \frac{A^2 + B^2 - 4C}{4}.$$

If the right-hand side is positive, *i.e.* if

$$A^2 + B^2 - 4C > 0,$$

then equation (1) represents a circle, whose center is at the point $(-\frac{1}{2}A, -\frac{1}{2}B)$ and whose radius is

$$\rho = \tfrac{1}{2}\sqrt{A^2 + B^2 - 4C}.$$

If, however, $A^2 + B^2 - 4C = 0$, then equation (1) represents just one point, namely, $(-\frac{1}{2}A, -\frac{1}{2}B)$, — or, if one prefers, a circle of zero radius or a null circle.

Finally, when $A^2 + B^2 - 4C < 0$, there are no points whose coördinates satisfy (1). To sum up, then:

Equation (1) *represents a circle, a single point, or there is no point whose coördinates satisfy* (1), *according as the expression*

$$A^2 + B^2 - 4C$$

is positive, zero, or negative.

Consider, more generally, the equation

(6) $$a(x^2 + y^2) + bx + cy + d = 0.$$

If $a = 0$, but b and c are not both 0, the equation represents a straight line.

If, however, $a \neq 0$, the equation can be divided through by a, and it thus takes on the form:

$$x^2 + y^2 + \frac{b}{a}x + \frac{c}{a}y + \frac{d}{a} = 0.$$

This is precisely the form of equation (1), and hence the above discussion is applicable to it.

EXERCISES

Determine what the following equations represent. Apply each time the *method* of completing the square and examining the right-hand side of the new equation. Do *not* merely substitute numerical values in the formulas developed in the text.

1. $x^2 + y^2 + 6x - 8y = 0$.

Ans. A circle, radius 5, with center at $(-3, 4)$.

2. $x^2 + y^2 - 6x + 4y + 13 = 0$. *Ans.* The point $(3, -2)$.

3. $x^2 + y^2 + 2x + 4y + 6 = 0$. *Ans.* No point whatever.

4. $x^2 + y^2 - 10x + 24y = 0$.

5. $x^2 + y^2 - 7x = 5$.

6. $x^2 + y^2 - 6x + 8y + 25 = 0$.

7. $49x^2 + 49y^2 - 14x + 28y + 5 = 0$.

Ans. The point $(\frac{7}{49}, -\frac{14}{49})$.

8. $x^2 + y^2 + 8y = 10$.

9. $x^2 + y^2 = 2ax$.

10. $x^2 + y^2 = 2ay$.

11. $x^2 + y^2 - 6ax - 2by + 9a^2 = 0$.

12. $x^2 + y^2 + 4ax - 8by + 16b^2 = 0$.

13. $x^2 + y^2 + 3 = 0$.

14. $x^2 + y^2 - 2x + 4y + 10 = 0$.

15. $3x^2 + 3y^2 - 4x + 2y + 7 = 0$.

16. $5x^2 + 5y^2 - 6x + 8y = 12$.

17. $3x^2 + 3y^2 - x + y = 6$.

3. Tangents. Let the circle

(1) $$x^2 + y^2 = \rho^2$$

be given, and let $P_1:(x_1, y_1)$ be any point of this circle. To find the equation of the tangent at P_1.

The tangent at P_1 is, by Elementary Geometry, perpendicular to the radius, OP_1. Hence its slope, λ', is the negative reciprocal of the slope, y_1/x_1, of OP_1; or

$$\lambda' = -\frac{x_1}{y_1}.$$

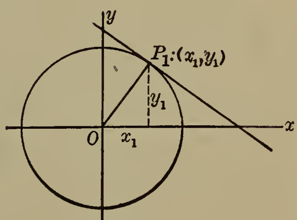

FIG. 2

We wish, therefore, to find the equation of the line which passes through the point (x_1, y_1) and has the slope $\lambda' = -x_1/y_1$. By Ch. II, § 2, (II), the equation of this line is

(2) $$y - y_1 = -\frac{x_1}{y_1}(x - x_1).$$

This equation can be simplified by multiplying through by y_1 and transposing:

(3) $$x_1 x + y_1 y = x_1^2 + y_1^2.$$

Now, the point (x_1, y_1) is, by hypothesis, on the circle; hence its coördinates satisfy the equation (1) of the circle:

$$x_1^2 + y_1^2 = \rho^2.$$

The right-hand side of equation (3) can, therefore, be replaced by the simpler expression, ρ^2.

We thus obtain, as the final form of the equation of the tangent, the following:

(4) $$x_1 x + y_1 y = \rho^2.$$

In deducing this equation it was tacitly assumed that $y_1 \neq 0$, since otherwise we could not have divided by it in obtaining λ'. The final formula, (4), is true, however, even when $y_1 = 0$, as can be directly verified. For, if $y_1 = 0$ and $x_1 = \rho$, then (4) becomes

$$\rho x = \rho^2 \qquad \text{or} \qquad x = \rho,$$

and this is the equation of the tangent in the point $(\rho, 0)$. Similarly, when $y_1 = 0$ and $x_1 = -\rho$.

Any Circle. If the given circle is represented by the equation

(5) $$(x - \alpha)^2 + (y - \beta)^2 = \rho^2,$$

precisely the same reasoning can be applied. The equation of the tangent to (5) at the point $P_1 : (x_1, y_1)$ of that circle is thus found to be:

(6) $$(x_1 - \alpha)(x - \alpha) + (y_1 - \beta)(y - \beta) = \rho^2.$$

The proof is left to the student as an exercise.

If the equation of the circle is given in the form

(7) $$x^2 + y^2 + Ax + By + C = 0,$$

or in the form (6), § 2, the equation can first be thrown into the form (5), and then the equation of the tangent is given by (6).

Example. To find the equation of the tangent to the circle

(8) $$3x^2 + 3y^2 + 8x - 5y = 0$$

at the origin.

First, reduce the coefficients of the terms in x^2 and y^2 to unity:

$$x^2 + y^2 + \tfrac{8}{3}x - \tfrac{5}{3}y = 0.$$

Next, complete the squares:

$$x^2 + \tfrac{8}{3}x + (\tfrac{4}{3})^2 + y^2 - \tfrac{5}{3}y + (\tfrac{5}{6})^2 = \tfrac{16}{9} + \tfrac{25}{36} = \tfrac{89}{16},$$

or $$(x + \tfrac{4}{3})^2 + (y - \tfrac{5}{6})^2 = \tfrac{89}{16}.$$

Now, apply the theorem embodied in formula (6). Since

$$x_1 = 0, \quad y_1 = 0, \quad \alpha = -\tfrac{4}{3}, \quad \beta = \tfrac{5}{6},$$

we have $$\tfrac{4}{3}(x + \tfrac{4}{3}) - \tfrac{5}{6}(y - \tfrac{5}{6}) = \tfrac{89}{16},$$

or $$8x - 5y = 0,$$

as the equation of the tangent to (8) at the origin.

EXERCISES

Find the equation to the tangent of each of the following circles at the given point.

1. $x^2 + y^2 = 25$ at $(-3, 4)$. *Ans.* $3x - 4y + 25 = 0$.
2. $x^2 + y^2 = a^2$ at $(0, a)$. *Ans.* $y = a$.
3. $x^2 + y^2 = 49$ at $(-7, 0)$.
4. $(x - 1)^2 + (y + 2)^2 = 25$ at $(4, 2)$. *Ans.* $3x + 4y = 20$.
5. $(x + 5)^2 + (y - 3)^2 = 49$ at $(2, 3)$.
6. $x^2 + y^2 - 9x + 11y = 0$ at the origin.

7. $2x^2 + 2y^2 - 3x - y = 11$ at $(-1, 2)$.

8. Find the intercepts on the axis of x made by the tangent at $(-5, 12)$ to

$$x^2 + y^2 = 169. \qquad \textit{Ans.} \quad -33\tfrac{4}{5}.$$

9. Find the area of the triangle cut from the first quadrant by the tangent at (1, 1) to

$$3x^2 + 3y^2 + 8x + 16y = 30.$$

10. If the equation

$$x^2 + y^2 + Ax + By + C = 0$$

represents a circle, and if the point (x_1, y_1) lies on the circle, show that the equation of the tangent at this point can be written in the form:

$$(9) \qquad x_1x + y_1y + \frac{A}{2}(x + x_1) + \frac{B}{2}(y + y_1) + C = 0.$$

Suggestion. Find the values of α, β, and ρ for the circle, substitute them in (6), and simplify the result.

11. Do Exs. 6 and 7, using formula (9), Ex. 11.

12. The same for the tangent to the circle in Ex. 9.

13. Show that, if $P_1 : (x_1, y_1)$ is any point of the circle

$$x^2 + y^2 + Ax + By + C = 0,$$

at which the tangent is not parallel to the axis of y, then the slope of the tangent at P_1 is

$$-\frac{2x_1 + A}{2y_1 + B}.$$

4. Circle through Three Points. It is shown in Elementary Geometry that a circle can be passed through any three points not lying in a straight line.

If the points are (x_1, y_1), (x_2, y_2), and (x_3, y_3), and if the equation of the circle through them is written in the form

$$x^2 + y^2 + Ax + By + C = 0,$$

then clearly the following three equations must hold:

$$x_1^2 + y_1^2 + Ax_1 + By_1 + C = 0,$$
$$x_2^2 + y_2^2 + Ax_2 + By_2 + C = 0,$$
$$x_3^2 + y_3^2 + Ax_3 + By_3 + C = 0.$$

We thus have three simultaneous linear equations for determining the three unknown coefficients A, B, C.

Suppose, for example, that the given points are the following:

$$(1, 1), \qquad (1, -1), \qquad (-2, 1).$$

The equations can be thrown at once into the form

$$A + B + C = -2,$$
$$A - B + C = -2,$$
$$-2A + B + C = -5.$$

Solve two of these equations for two of the unknowns in terms of the third. Then substitute the values thus found in the third equation. Thus the third unknown is completely determined, and hence the other two unknowns can be found.

Here, it is easy to solve the first two equations for A and B in terms of C. On subtracting the second equation from the first, we find:

$$2B = 0; \qquad \text{hence} \qquad B = 0.$$

Then either of the first two equations gives for A the value:

$$A = -C - 2.$$

Next, set for A and B in the third equation the values just found:

$$2C + 4 + C = -5, \qquad C = -3.$$

Hence, finally,

$$A = 1, \qquad B = 0, \qquad C = -3,$$

and the equation of the desired circle is:

$$x^2 + y^2 + x - 3 = 0.$$

Check the result by substituting the coördinates of the given points successively in this last equation. They are found each time to satisfy the equation.

The circle through the three given points has its center in the point $(-\frac{1}{2}, 0)$. Its radius is of length $\sqrt{3.25} = 1.803$.

EXERCISES

Find the equations of the circles through the following triples of points. Plot the points and draw the circles.

1. (1, 0), (0, 1), the origin. *Ans.* $x^2 + y^2 - x - y = 0$.

2. (1, 1), (– 1, – 1), (1, – 1).

3. (5, 10), (6, 9), (– 2, 3).

4. The vertices of the triangle of Ex. 15 at the end of Ch. III, p. 63. Show that the point (2, – 1) of that exercise lies on the circle.

5. The same question for Ex. 17, p. 63. Show that the point $\left(0, \frac{ab}{c}\right)$ of that exercise lies on the circle.

6. The vertices of the triangle of Ch. III, Fig. 1. Find the coördinates of the center and check by comparing them with those of the point of intersection of the perpendicular bisectors of the sides of the triangle, as determined in Ch. III, § 5.

7. The same question for the triangle of Ch. III, Fig. 2. Check.

8. The vertices of the triangle formed by the coördinate axes and the line $2x - 3y = 6$.

9. The vertices of the triangle whose sides are:

$$x - y - 1 = 0, \quad x + y + 2 = 0, \quad 2x - y + 3 = 0.$$

Ans. $3x^2 + 3y^2 + 17x + 16y + 25 = 0$.

EXERCISES ON CHAPTER IV

1. Find the equation of the circle with the line-segment joining the two points (3, 0) and (5, 2) as a diameter.

2. A circle goes through the origin and has intercepts – 5 and 3 on the axes of x and y respectively. Find its equation.

3. A circle goes through the origin and has intercepts a and b. Find its equation.

4. Find the equation of the circle which has its center in the point $(-3, 4)$ and is tangent to the line $3x + 8y - 6 = 0$.

5. A circle has its center on the line $2x - 3y = 0$ and passes through the points $(4, 3)$, $(-2, 5)$. Find its equation.

6. Find the equation of the circle which passes through the point $(5, -2)$ and is tangent to the line $3x - y - 1 = 0$ at the point $(1, 2)$.

7. There are two circles passing through the points $(3, 2)$, $(-1, 0)$ and having 6 as their radius. Find their equations.

8. There are two circles with their centers on the line, $5x - 3y = 8$, and tangent to the coördinate axes. Find their equations.

9. Find the equations of the circles tangent to the axes and passing through the point $(1, 2)$.

10. Find the equations of the circles passing through the points $(3, 1)$, $(1, 0)$ and tangent to the line $x - y = 0$.

Suggestion. Demand that the center (α, β) be equally distant from the two points and the line.

11. Find the equations of the circles passing through the origin, tangent to the line $x + y - 8 = 0$, and having their centers on the line $x = 2$.

12. Find the equations of the circles of the preceding exercise, if their centers lie on the line $2x - y - 2 = 0$.

13. Find the equation of the circle inscribed in the triangle formed by the axes and the line $3x - 4y - 12 = 0$.

14. Find the equation of an arbitrary circle, referred to two perpendicular tangents as axes.

15. Do the four points $(0, 0)$, $(6, 0)$, $(0, -4)$, $(5, 1)$ lie on a circle?

16. Find the coördinates of the points of intersection of the circles

$$x^2 + y^2 - x + 2y = 0,$$
$$x^2 + y^2 + 2x - y = 9.$$

17. Find the coördinates of the points of intersection of the circles

$$x^2 + y^2 + ax + by = 0,$$
$$x^2 + y^2 + bx - ay = 0.$$

Orthogonality

18. A circle and a line intersect in a point P. The acute angle between the line and the tangent to the circle at P is known as the angle of intersection of the line and the circle at P. If the line meets the circle in two points, the angles of intersection at the two points are equal. Determine the angle in the case of the circle

$$x^2 + y^2 = 25,$$

and the line
$$2x - y - 5 = 0.$$

19. A circle and a line are said to intersect orthogonally if their angle of intersection is a right angle. Prove that the circle,

$$x^2 + y^2 - 4x + 6y + 3 = 0,$$

is intersected by the line, $5x + y = 7$, orthogonally.

Suggestion. First answer geometrically the question: What lines cut a given circle orthogonally?

20. Show that the circle,

$$x^2 + y^2 + Ax + By + C = 0,$$

intersects the line,

$$ax + by + c = 0,$$

orthogonally when and only when

$$aA + bB = 2c.$$

21. If two circles intersect in a point P, the acute angle between their tangents at P is known as their angle of intersection. If the circles intersect in two points, their angles of intersection at these points are equal. Find this angle in the case of the circles,

$$x^2 + y^2 = 25,$$
$$x^2 + y^2 - 7x + y = 0.$$

22. Prove geometrically that two circles intersect orthogonally, that is, at right angles, when and only when the sum of the squares of their radii equals the square of the distance between their centers. Then show that the circles

$$x^2 + y^2 - 4x + 5y - 2 = 0,$$
$$2x^2 + 2y^2 + 4x - 6y - 19 = 0,$$

intersect orthogonally.

23. Prove that the two circles,

$$x^2 + y^2 + A_1x + B_1y + C_1 = 0,$$
$$x^2 + y^2 + A_2x + B_2y + C_2 = 0,$$

intersect orthogonally when and only when

$$A_1A_2 + B_1B_2 = 2C_1 + 2C_2.$$

24. Find the equation of the circle which cuts the circle

$$x^2 + y^2 + 2x = 0$$

at right angles and passes through the points (1, 0) and (0, 1).

25. There are an infinite number of circles cutting each of the two circles,

$$x^2 + y^2 - 4y + 2 = 0,$$
$$x^2 + y^2 + 4y + 2 = 0,$$

orthogonally. Show that they are all given by the equation

$$x^2 + y^2 + ax - 2 = 0,$$

where a is an arbitrary constant. Where are their centers? Draw a figure.

26. Find the equation of the circle cutting orthogonally the three circles,

$$x^2 + y^2 = 9,$$
$$x^2 + y^2 + 3x - 5y + 6 = 0,$$
$$x^2 + y^2 - 2x + 3y - 19 = 0.$$

Ans. $x^2 + y^2 + 10x + 9 = 0.$

Miscellaneous Theorems

27. Prove analytically that every angle inscribed in a semicircle is a right angle.

28. Prove analytically that the perpendicular dropped from a point of a circle on a diameter is a mean proportional between the segments in which it divides the diameter.

29. The tangents to a circle at two points P, Q meet in the point T. The lines joining P and Q to one extremity of the diameter parallel to PQ meet the perpendicular diameter in the points R and S. Prove that $RT = ST$.

30. In a triangle the circle through the mid-points of the sides passes through the feet of the altitudes and also through the points halfway between the vertices and the point of intersection of the altitudes. This circle is known as the *Nine-Point Circle* of the triangle.

For the triangle with vertices in the points $(-4, 0)$, $(2, 0)$, $(0, 6)$ construct the circle and mark the nine points through which it passes.

31. For the triangle in the preceding exercise find the equation of the nine-point circle, as the circle through the mid-points of the sides. *Ans.* $3x^2 + 3y^2 + 3x - 11y = 0$.

32. Show that this circle goes through the other six points.

33. For the triangle with vertices in the points $(a, 0)$, $(b, 0)$, $(0, c)$ find the equation of the nine-point circle, as the circle through the mid-points of the sides.

Ans. $2c(x^2 + y^2) - (a + b)cx + (ab - c^2)y = 0$.

34. Show that this circle goes through the other six points.

CHAPTER V

INTRODUCTORY PROBLEMS IN LOCI. SYMMETRY OF CURVES

1. Locus Problems.* *A point is moving under given conditions; its locus is required.* This type of problem the student studied in Plane Geometry. But he found there no general method, by means of which he could always determine a locus; for each problem he had to devise a method, depending on the particular conditions of the problem.

Analytic Geometry, however, provides a general method for the determination of loci. Some simple examples of the method have already been given. Thus, in finding the equation of a circle, we determined the locus of a point whose distance from a fixed point is constant. Again, in deducing the equation of a line through two points, we found the locus of a point moving so that the line joining it to a given point has a given direction.

The method in each of these cases consisted merely in expressing in analytic terms — *i.e.* in the form of an equation involving the variable coördinates, x and y, of the moving point — the given geometric condition under which the point moved. We proceed to show how this method applies in less simple cases.

Example 1. The base of a triangle is fixed, and the distance from one end of the base to the mid-point of the opposite side is given. Find the locus of the vertex.

* The locus problems in this chapter may be supplemented, if it is desired, by §§ 6–8 of the second chapter on loci, Ch. XIII, in which the loci of inequalities and the bisectors of the angles between two lines, together with related subjects, are considered.

Let the triangle be OAP, with M as the mid-point of AP. Let a be the length of the base OA, and let l be the given distance. It is required to find the locus of P, so that always

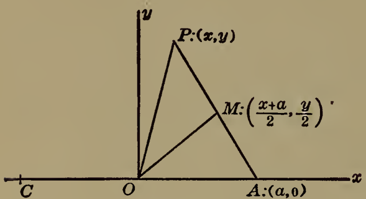

FIG. 1

(1) $$OM = l.$$

It is convenient to take the origin of coördinates in O and the positive axis of x along the base. The coördinates of A are then $(a, 0)$. The coördinates of the moving point P we denote by (x, y). The coördinates of the point M are

$$\left(\frac{x+a}{2}, \frac{y}{2}\right).$$

The distance OM is

$$\sqrt{\left(\frac{x+a}{2}\right)^2 + \left(\frac{y}{2}\right)^2}.$$

Thus condition (1), expressed analytically, is

$$\sqrt{\left(\frac{x+a}{2}\right)^2 + \left(\frac{y}{2}\right)^2} = l.$$

Squaring both sides of this equation and simplifying, we have

(2) $$(x+a)^2 + y^2 = (2\,l)^2.$$

This equation represents the circle whose center is at $(-a, 0)$ and whose radius is $2\,l$. We have shown, therefore, that, if (1) is always satisfied, the coördinates (x, y) of P satisfy (2), and P lies on the circle. The locus of P appears, then, to be the circle.

How do we know, though, that P traces the entire circle? To prove this, we must show, conversely, that, if the coördinates (x, y) of P satisfy (2), condition (1) is valid. If (x, y) satisfy (2), then, on dividing both sides of (2) by 4 and extracting the square root of each side, we obtain *two* equations:

i) $$\sqrt{\left(\frac{x+a}{2}\right)^2+\left(\frac{y}{2}\right)^2}=l,$$

ii) $$\sqrt{\left(\frac{x+a}{2}\right)^2+\left(\frac{y}{2}\right)^2}=-l.$$

Equation ii) says that a positive or zero quantity equals a negative quantity, and is therefore impossible. Thus only equation i) remains. This equation says that $OM=l$. Hence condition (1) is satisfied by every point of the circle,* and so the circle is the locus of P.

We have yet to describe the locus, independently of the coördinate system, with reference merely to the original triangle. Produce the base, in the direction from A to O, to the point C, doubling its length. Then the locus of P is a circle, whose center is at C and whose radius is twice the given distance.

Example 2. Determine the locus of a point P which moves so that the difference of the squares of its distances from two fixed points P_1, P_2 is constant, and equal to c:

$$(3)\quad \begin{cases} PP_1^2-PP_2^2=c, \\ \quad\text{or} \\ PP_2^2-PP_1^2=c. \end{cases}$$

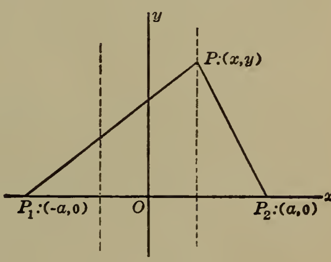

Fig. 2

Take the mid-point of the segment P_1P_2 as origin and the axis of x along P_1P_2. The coördinates of P_1 and P_2 can be written as $(-a, 0)$, $(a, 0)$; those of P, as (x, y). By Ch. I, § 3,

$$PP_1^2=(x+a)^2+y^2, \qquad PP_2^2=(x-a)^2+y^2.$$

Then the equations (3), expressed analytically, are

*The two points in which the circle cuts the axis of x are exceptions, since these do not lead to a triangle, OAP.

$$(x + a)^2 + y^2 - (x - a)^2 - y^2 = c,$$
$$(x - a)^2 + y^2 - (x + a)^2 - y^2 = c.$$

These reduce to

(4) $$4\,ax = c, \qquad 4\,ax = -\,c.$$

Hence, if condition (3) is satisfied, P lies on one or the other of the lines

(5) $$x = \frac{c}{4\,a}, \qquad x = -\frac{c}{4\,a}.$$

Conversely, if P lies on one or the other of the lines (5), then (4) holds, and from (4) we show by retracing the steps that one or the other of the equations (3) is valid.

Consequently, the locus of P consists of *two* straight lines, perpendicular to the line P_1P_2, and symmetrically situated with reference to the mid-point of P_1P_2, the distance of either line from the mid-point being $c/4\,a$. Thus the locus consists of two entirely unconnected pieces, one corresponding to each of the equations (3). If $c = 0$, these equations are the same, and the two lines forming the locus coincide in the perpendicular bisector of the segment P_1P_2.

EXERCISES

In solving the following problems, the first step is to find the equation of a curve, — or the equations of curves, — on which points of the locus lie. The student must then take care (*a*) to show, conversely, that every point lying on the curve or curves obtained satisfies the given conditions; and (*b*) to describe the locus, finally, without reference to the coördinate system used.

1. A point P moves so that the sum of the squares of its distances to two fixed points P_1, P_2 is a constant, c, greater than $\frac{1}{2}\,P_1P_2^{\,2}$. Show that the locus of P is a circle, with its center at the mid-point of P_1P_2.

What is the locus if $c = \frac{1}{2}\,P_1P_2^{\,2}$? If $c < \frac{1}{2}\,P_1P_2^{\,2}$?

2. Find the locus of the mid-point of a line of fixed length which moves so that its end points always lie on two mutually perpendicular lines.

3. Determine the locus of a point which moves so that the sum of the squares of its distances to the sides, or the sides produced, of a given square is constant. Is there any restriction necessary on the value of the constant?

4. Determine the locus of a point which moves so that the square of its distance to the origin equals the sum of its coördinates. *Ans.* A circle, center at $(\frac{1}{2}, \frac{1}{2})$, radius $= \frac{1}{4}\sqrt{2}$.

5. Show that the locus of a point which moves so that the sum of its distances to two mutually perpendicular lines equals the square of its distance to their point of intersection consists of the arcs of four circles, forming a continuous curve. Where are the circles, and which of their arcs belong to the locus?

6. The base of a triangle is fixed, and the trigonometric tangent of one base angle is a constant multiple, not unity, of the trigonometric tangent of the other. Find the locus of the vertex.

2. Symmetry. In the problems of the preceding paragraph, the equations of the loci were familiar and the curves they represented were easily identified. In subsequent chapters, however, we shall have locus problems to consider in which the resulting equations will be new to us. In drawing the curves which these equations represent, it will be useful to have at hand the salient facts concerning the symmetry of curves.

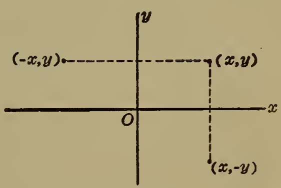

FIG. 3

Symmetry in a Line. Two points, P and P', are said to be symmetric in a line L, if L is the perpendicular bisector of PP'.

If L is the axis of x and (x, y) are the coördinates of P, then it is clear that $(x, -y)$ are the coördinates of P'.

Similarly, if L is the axis of y and P has the coördinates (x, y), then P' has the coördinates $(-x, y)$.

Example 1. Given the curve

(1) $$y^2 = x.$$

Let $P:(x_1, y_1)$ be any point on it, *i.e.* let

(2) $$y_1^2 = x_1$$

be a true equation. Then the point $P':(x_1, -y_1)$, symmetric to P in the axis of x, also lies on the curve. For, if we substitute the coördinates of P' into (1), the result is $(-y_1)^2 = x_1$, or (2), and (2) we know is a true equation. We say, then, that the curve (1) is symmetric in the axis of x.

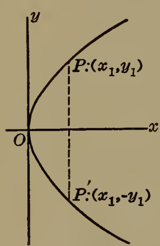

FIG. 4

The test for symmetry in the axis of x, employed in this example, is general in application. We state it, and the corresponding test for symmetry in the axis of y, in the form of theorems.

THEOREM 1. *A curve is symmetric in the axis of x if the substitution of $-y$ for y in its equation leaves the equation unchanged.*

THEOREM 2. *A curve is symmetric in the axis of y if the substitution of $-x$ for x in its equation leaves the equation unchanged.*

Symmetry in a Point. Two points, P and P', are symmetric in a given point, if the given point is the mid-point of PP'. If the given point is the origin of coördinates and P has the coördinates (x, y), then the coördinates of P' are evidently $(-x, -y)$.

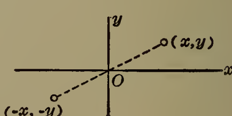

FIG. 5

Example 2. Consider the curve

(3) $$y = x^3.$$

If $P:(x_1, y_1)$ is any point on this curve, then the point $P':(-x_1, -y_1)$, symmetric to P in the origin, is also on the curve. For, the condition that P' lies on the curve, namely,

$$-y_1 = (-x_1)^3 \quad \text{or} \quad -y_1 = -x_1^3,$$

is equivalent to the condition: $y_1 = x_1^3$, that P lie on the curve. We say, then, that the curve (3) is symmetric in the origin.

This test, too, is general in application; we formulate it as a theorem.

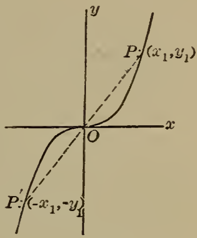

Fig. 6

THEOREM 3. *A curve is symmetric in the origin of coördinates, if the substitution of $-x$ for x, and of $-y$ for y, in its equation leaves the equation essentially unchanged.*

A case in which the test leaves the equation wholly unchanged is that of the circle, $x^2 + y^2 = \rho^2$, or the curve $xy = a^2$ (Fig. 7).

Now the circle in question is symmetric in both axes. It follows then, without further investigation, that it is symmetric in the origin, the point of intersection of the axes. This conclusion holds always; in fact, we may state the theorem.

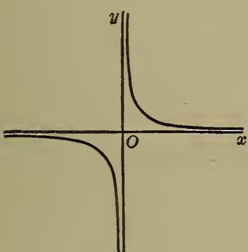

Fig. 7

THEOREM 4. *If a curve is symmetric in both axes of coördinates, it is symmetric in the origin.*

The details of the proof are left to the student as an exercise. It is to be noted that the converse of the theorem, namely, that if a curve is symmetric in the origin, it is symmetric in the axes, is not true. For, the curve of Example 2 is symmetric in the origin, but not symmetric in either axis; this is true also of the curve $xy = a^2$ of Fig. 7.

EXERCISES

1. Prove Theorem 4.

2. Test, for symmetry in each axis and in the origin, the curves given in the following exercises of Ch. I, § 7:

(*a*) Exercise 2; (*c*) Exercise 7;
(*b*) Exercise 6; (*d*) Exercise 8.

In each of the following exercises test the given curve for symmetry in each axis and in the origin. Plot the curve.

3. $xy + 1 = 0$.

4. $10y = x^4$.

5. $20x = y^5$.

6. $y^2 + 4x = 0$.

7. $x^2 - y^2 = 4$.

8. $x^2 + 2y^2 = 16$.

EXERCISES ON CHAPTER V

1. The base of a triangle is fixed and the ratio of the lengths of the two sides is constant. Find the locus of the vertex. *Ans.* A circle, except for one value of the constant.

2. A point P moves so that its distance from a given line L is proportional to the square of its distance to a given point K, not on L. If P remains always on the same side of L as K, show that its locus is a circle.

3. Find the locus of P in the preceding exercise, if it remains always on the opposite side of L from K. Does your answer cover all cases?

4. If, in Ex. 2, K lies on L and P may be on either side of L, what is the locus of P?

5. Three vertices of a quadrilateral are fixed. Find the locus of the fourth, if the area of the quadrilateral is constant.

6. Find the locus of a point moving so that the sum of the squares of its distances from the sides of an equilateral triangle is constant. Discuss all cases.

Ans. A circle, center at the point of intersection of the medians; this point; or no locus.

7. The feet of the perpendiculars from the point $P: (X, Y)$ on the sides of the triangle with vertices in the points (0, 0), (3, 0), (0, 1) lie on a line. Find the locus of P.

Ans. The circle circumscribing the triangle.

8. The preceding problem, if the triangle has the points (2, 0), (– 3, 0), (0, 4) as vertices.

9. Problem 7, for the general triangle, with vertices at $(a, 0)$, $(b, 0)$, $(0, c)$.

10. Show that the equation of the circle described on the line-segment joining the points (x_1, y_1), (x_2, y_2) as a diameter may be written in the form

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$$

Suggestion. Find the locus of a point P moving so that the two given points always subtend at P a right angle.

11. The two points, P and P', are symmetric in the line, $x - y = 0$, bisecting the angle between the positive axes of x and y. Show that, if (x, y) are the coördinates of P, then (y, x) are the coördinates of P'.

12. Prove that a curve is symmetric in the line $x - y = 0$ if the interchange of x and y in its equation leaves the equation unchanged.

13. If P and P' are symmetric in the line $x + y = 0$ and P has the coördinates (x, y), show that the coördinates of P' are $(-y, -x)$.

14. Give a test for the symmetry of a curve in the line $x + y = 0$.

15. Test each of the following curves for symmetry in the lines $x - y = 0$ and $x + y = 0$.

(*a*) $xy = a^2$; (*c*) $x^2 - y^2 = a^2$;

(*b*) $xy = -a^2$; (*d*) $(x - y)^2 - 2x - 2y = 0$.

16. Plot the curve of Ex. 15, (*d*).

In each of the following exercises find the equation of the locus of the point P. Plot the locus from the equation, making all the use possible of the theory of symmetry.

17. The distance of P from the line $x - 2 = 0$ equals its distance from the point $(2, 0)$.

18. The sum of the distances of P from the points $(3, 0)$ and $(-3, 0)$ is 10.

19. The difference of the distances of P from the points $(5, 0)$ and $(-5, 0)$ is 8.

CHAPTER VI

THE PARABOLA

1. Definition. A *parabola* is defined as the locus of a point P, whose distance from a fixed line D is always equal to its distance from a fixed point F, not on the line. It is understood, of course, that P is restricted to the plane determined by D and F.

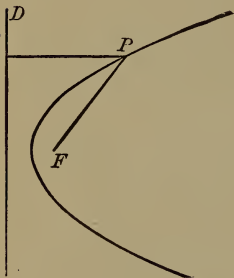

FIG. 1

One point of the locus is the mid-point A, Fig. 2, of the perpendicular FE dropped from F on D. Through A draw T parallel to D. Then no other point on T, or to the left of T, can belong to the locus, for all such points are clearly nearer to D than they are to F.

Further points of the locus can be obtained as follows. To the right of T draw L parallel to D, cutting AF, produced if necessary, in S. With ES as radius and F as center describe a circle, cutting L in P and Q. Then P and Q lie on the locus.

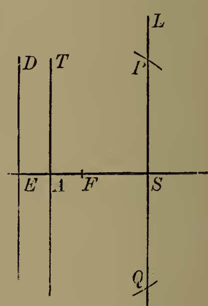

FIG. 2

A large number of points having been obtained in this way, a smooth curve can be passed through them. The curve is symmetric in the line AF, and evidently has T as a tangent.

The line D is called the *directrix*, and the point F, the *focus*, of the parabola; A is the *vertex*, and the indefinite line AF, the *axis*; FP is a *focal radius*.

The student is familiar with the fact that all circles are *similar*; *i.e.* have the same shape, and differ only in size. A like relation holds for any two parabolas. Think of them as lying in different planes, and choose in each plane as the unit length the distance between the focus and the directrix. Then the one parabola, in its plane, is the replica of the other, in its plane. Consequently, the two parabolas differ only in the scale to which they are drawn, and are, therefore, similar.

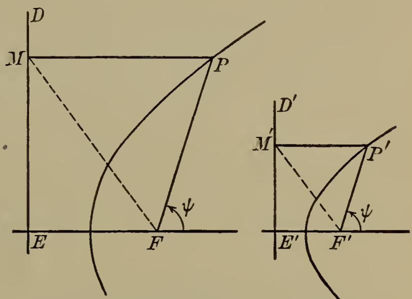

Fig. 3

The details of the proof just outlined can be supplied at once by showing that the triangles FPM and MFE are similar, respectively, to $F'P'M'$ and $M'F'E'$, the angles ψ in Fig. 3 being equal by construction. Hence

$$\frac{FP}{F'P'} = \frac{EF}{E'F'},$$

i.e. focal radii, FP and $F'P'$, which make the same angle with the axes always bear to each other the same fixed ratio.

EXERCISES

1. Take a sheet of squared paper and mark D along one of the vertical rulings near the edge of the paper. Choose F at a distance of 1 cm. from D. Then the points of the locus on the vertical rulings — or on as many of them as one desires — can be marked off rapidly with the compasses. Make a clean, neat figure.

2. Place a card under the curve of Ex. 1 and, with a needle, prick numerous points of the curve through on the card, and mark, also, the focus and axis in this way. Cut the card along

the curve with sharp scissors. The piece whose edge is convex forms a convenient parabolic ruler, or templet, to be used whenever an accurate drawing is desired.

A small hole at the focus and a second hole farther along the axis make it possible, in using the templet, to mark the focus and draw the axis.

A second templet, to twice the above scale, will also be found useful.

3. The focus of a parabola is distant 5 units from the directrix. In a second parabola, this distance is 2 units. How much larger is the first parabola than the second, *i.e.*, how do their scales compare with each other?

2. Equation of the Parabola. The first step is to choose the axes of coördinates in a convenient manner. Evidently, one good choice would be to take the axis of x perpendicular to D and passing through F. Let us do this, choosing the positive sense from A toward F.

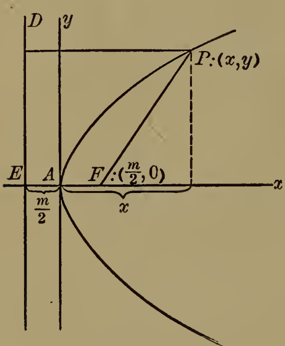

Fig. 4

For the axis of y three simple choices present themselves, namely:

(*a*) through A;
(*b*) along D;
(*c*) through F.

Perhaps (*b*) seems most natural; but (*a*) has the advantage that the curve then passes through the origin, and this choice turns out in practice to be the most useful one. We will begin with it.

Let $P:(x, y)$ be any point on the curve. Denote the distance of F from D by m. Then

$$EF = m, \qquad AF = \frac{m}{2}, \qquad \text{and} \qquad EA = \frac{m}{2}.$$

By Ch. I, § 3

$$FP=\sqrt{\left(x-\frac{m}{2}\right)^2+y^2}.$$

On the other hand, the distance of P from D is

$$x+\frac{m}{2}.$$

By definition, these two distances are equal, or:

$$(1) \qquad \sqrt{\left(x-\frac{m}{2}\right)^2+y^2}=x+\frac{m}{2}.$$

Square each side of the equation, so as to remove the radical, and expand the binomials:

$$(2) \qquad x^2-mx+\frac{m^2}{4}+y^2=x^2+mx+\frac{m^2}{4}.$$

The result can be reduced at once to the form

$$(3) \qquad y^2=2mx,$$

and this is the equation of the parabola, referred to its vertex as origin and to its axis as the axis of x.

The proof of this last statement is not yet, however, complete; for it remains to show conversely that, if (x, y) be any point whose coördinates satisfy (3), it is a point of the parabola. From (3) we can pass to (2). On extracting the square root of each side of (2), we have two equations:

$$\text{i)} \qquad \sqrt{\left(x-\frac{m}{2}\right)^2+y^2}=x+\frac{m}{2},$$

$$\text{ii)} \qquad \sqrt{\left(x-\frac{m}{2}\right)^2+y^2}=-\left(x+\frac{m}{2}\right),$$

one of which *must* be true, and both of which *may conceivably* be true. Now, x is a positive quantity or zero; for, by hypothesis, the coördinates of the point (x, y) satisfy equation (3). Hence ii) is impossible, for it says that a positive or zero quantity is equal to a negative quantity. Thus only i) remains, and this equation is precisely the condition that the distance of (x, y) from D be equal to its distance from F. Hence the point (x, y) lies on the parabola, q. e. d.

EXERCISES

1. Show that the choice (*b*) leads to the equation

$$y^2 = 2mx - m^2. \tag{4}$$

This is the equation of the parabola referred to its directrix and axis as the axes of y and x respectively, with the positive axis of x in the direction in which the curve opens.

2. Show that the choice (*c*) leads to the equation

$$y^2 = 2mx + m^2. \tag{5}$$

This is the equation of the parabola when the focus is the origin and the positive axis of x is along the axis of the curve in the direction in which the curve opens.

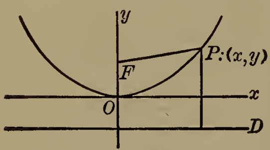

Fig. 5

3. Taking the axes as indicated in Fig. 5, show that the equation of the parabola is

$$x^2 = 2my.$$

4. Choosing the axis of y as in the foregoing question, show that the equation of the parabola is

$$x^2 = 2my - m^2,$$

in case the axis of x is along D, and is

$$x^2 = 2my + m^2,$$

in case F is taken as the origin.

5. If the axis of x is taken along the axis of the parabola, but positively in the direction from F toward D, and if the origin is taken at the vertex, show that the equation of the curve is

$$y^2 = -2mx.$$

6. If the axis of y is taken along the axis of the parabola, but positively in the direction from F toward D, and if the origin is taken at the vertex, show that the equation of the curve is

$$x^2 = -2my.$$

7. Determine the focus and directrix of each of the following parabolas:

(a) $y^2 = 4x$. *Ans.* $(1, 0)$; $x + 1 = 0$.

(b) $y = x^2$. *Ans.* $(0, \frac{1}{4})$; $4y + 1 = 0$.

(c) $3y^2 - 5x = 0$. (d) $3y^2 + 22x = 0$.

(e) $y = -2x^2$. (f) $5x^2 + 12y = 0$

(g) $y^2 = px$. (h) $x^2 = 4ay$.

8. It appears from the foregoing that any equation of the form

$$y^2 = \pm Ax, \qquad \text{or} \qquad x^2 = \pm Ay,$$

where A is any positive constant, represents a parabola with its vertex at the origin. Formulate a general rule for ascertaining the distance of the focus of such a parabola from the vertex.

9. Find the equations of the following parabolas:

(a) Vertex at $(0, 0)$ and focus at $(2, 0)$.

(b) Vertex at $(0, 0)$ and $2x + 5 = 0$ as directrix.

(c) Vertex at $(0, 0)$ and focus at $(0, -\frac{3}{5})$.

(d) Vertex at $(0, 0)$ and $2y - 1 = 0$ as directrix.

(e) Focus at $(0, 0)$ and vertex at $(-3, 0)$.

(f) Focus at $(0, 0)$ and $3y + 4 = 0$ as directrix.

(g) Focus at $(6, 0)$ and axis of y as directrix.

(h) Focus at $(0, -7)$ and axis of x as directrix.

3. **Tangents.** The student will next turn to Chapter IX and study §§ 1, 2. It is there shown that the slope of the parabola

(1) $$y^2 = 2mx$$

at any one of its points (x_1, y_1) is, in general, given by the formula

(2) $$\lambda = \frac{m}{y_1};$$

and that the equation of the tangent line at any point (x_1, y_1) can, without exception, be written in the form

(3) $$y_1 y = m(x + x_1).$$

Latus Rectum. The chord, PP', of a parabola which passes through the focus and is perpendicular to the axis is called the *latus rectum* (plural, *latera recta*).

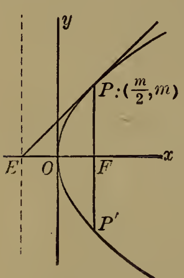

Fig. 6

Its half-length is found by setting $x = m/2$ in the equation of the parabola, and solving for the positive y:

$$y^2 = 2m\left(\frac{m}{2}\right) = m^2, \qquad y = m.$$

Thus the length, PP', of the latus rectum is $2m$.

The tangent at either P or P' makes an angle of 45° with the axis of x. For, the slope of the tangent at P is, from (2):

$$\frac{m}{y_1} = \frac{m}{m} = 1.$$

Let E be the point in which the tangent at P meets the axis of x. Since $FP = m$, and $\angle FEP = 45°$, $EF = m$ and so E lies on the directrix. Consequently, the tangents at P and P' cut the axis of x at the point of intersection of the directrix with that axis.

This theorem can also be proved by writing down the equation of the tangent at P,

$$y = x + \frac{m}{2},$$

and finding the intercept of this line on the axis of x.

EXERCISES

1. Find the equation of the tangent to the parabola $y^2 = 3x$ at the point (12, 6). *Ans.* $x - 4y + 12 = 0$.

2. Find the equation of the normal to the same parabola at the given point. *Ans.* $4x + y = 54$.

3. Find the length of the latus rectum of the parabola of Ex. 1.

4. Show that the tangents to any parabola at the extremities of the latus rectum are perpendicular to each other.

5. Show that the tangent to the parabola $y^2 = 4x$ at the point (36, 12) cuts the negative axis of x at a point whose distance from the origin is 36.

6. At what point of the parabola of Ex. 5 is the tangent perpendicular to the tangent mentioned in that exercise?
Ans. $(\frac{1}{36}, -\frac{1}{3})$.

7. Show that the two tangents mentioned in Exs. 5 and 6 intersect on the directrix, and that the *chord of contact* of these tangents, *i.e.* the right line drawn through the two points of tangency, passes through the focus.

8. Show that the tangent to the parabola (1) at any point P cuts the negative axis of x at a point M whose distance from the origin is the same as the distance of P from the axis of y.

9. Prove that the two parabolas,

$$y^2 = 4x + 4 \qquad \text{and} \qquad y^2 = -6x + 9,$$

intersect at right angles. Assume that the slope of the parabola of Ex. 2, § 2, at the point (x_1, y_1) is m/y_1.

10. If two parabolas have a common focus and their axes lie along the same straight line, their vertices, however, being on opposite sides of the focus, show that the curves cut each other at right angles.

4. Optical Property of the Parabola. If a polished reflector, like the reflector of the headlight of a locomotive or a searchlight, be made in the form of a paraboloid of revolution, *i.e.* the surface generated by a parabola which is revolved about its axis, and if a source of light be placed at the focus, the reflected rays will all be parallel.

This phenomenon is due to the fact that the focal radius FP drawn to any point P of the parabola makes the same angle with the tangent at P as does the line through P parallel to the axis.

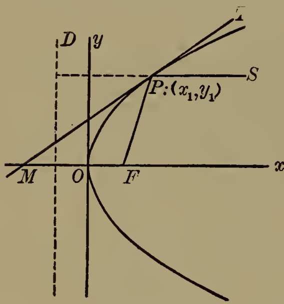

FIG. 7

The proof of this property can be given as follows. Let the tangent at $P:(x_1, y_1)$ cut the axis of x in M. Then the length of OM is equal to x_1, by § 3, Ex. 8. Furthermore, $OF = m/2$. Hence the distance from M to F is

$$MF = x_1 + \frac{m}{2}.$$

But this is precisely the distance of P from D, § 2, and hence, by the definition of the parabola, it is also equal to FP. We have, then, that $MF = FP$. Consequently, the triangle MFP is isosceles, and

$$\measuredangle FMP = \measuredangle MPF.$$

But
$$\measuredangle FMP = \measuredangle SPT,$$

and the proposition is proved.

The result can be restated in the following

THEOREM. *The focal radius FP of a parabola at any point P of the curve and the parallel to the axis at P make equal angles with the tangent at P.*

Heat. If such a parabolic reflector as the one described above were turned toward the sun, the latter's rays, being practically parallel to each other and to the axis of the reflector, would, after impinging on the polished surface, proceed along lines, all of which would pass through F. Thus, in particular, the heat rays would be collected at F, and if a minute charge of gunpowder were placed at F, it might easily be fired.

It is to this property that the *focus* (German, *Brennpunkt*) owes its name. The Latin word means *hearth*, or *fireplace*. The term was introduced into the science by the astronomer Kepler in 1604.

EXERCISES ON CHAPTER VI

1. A parabola opens out along the positive axis of y as axis. Its focus is in the point (0, 3) and the length of its latus rectum is 12. Find its equation. *Ans.* $x^2 = 12y$.

2. A parabola has its vertex in the origin and its axis along the axis of x. If it goes through the point $(2, -3)$, what is its equation? *Ans.* $2y^2 - 9x = 0$.

3. Show that the equation of a parabola with the line $x = c$ as directrix and with the point $(c + m, 0)$ or $(c - m, 0)$ as focus is

$$y^2 = 2m(x - c) - m^2, \qquad \text{or} \qquad y^2 = -2m(x - c) - m^2.$$

Hence prove that every parabola with the axis of x as axis has an equation of the form: $x = ay^2 + b$, where a and b are constants, $a \neq 0$.

4. Find the equation of the parabola which has its axis along the axis of x and goes through the two points (3, 2), $(-2, -1)$. *Ans.* $3x = 5y^2 - 11$.

5. Prove that every parabola with an axis parallel to the axis of y has an equation of the form

$$y = ax^2 + bx + c,$$

where a, b, c are constants, $a \neq 0$.

Suggestion. Find the equation of the parabola which has the line $y = k$ as directrix and the point $(l, k + m)$ or $(l, k - m)$ as focus.

6. Find the equation of the parabola which has a vertical axis and goes through the points (0, 0), (1, 0), and (3, 6). *Ans.* $y = x^2 - x$.

7. A circle is tangent to the parabola $y^2 = x$ at the point (4, 2) and goes through the vertex of the parabola. Find its equation.

8. What is the equation of the circle which is tangent to the parabola $y^2 = 2mx$ at both extremities of the latus rectum?

Ans. $4x^2 + 4y^2 - 12mx + m^2 = 0$.

9. Find the coördinates of the points of tangency of the tangents to the parabola $y^2 = 2mx$ which make the angles 60°, 45°, and 30° with the axis of the parabola. Show that the abscissæ of the three points are in geometric progression, and that this is true also of the ordinates.

10. Show that the common chord of a parabola, and the circle whose center is in the vertex of the parabola and whose radius is equal to three halves the distance from the vertex to the focus, bisects the line-segment joining the vertex with the focus.

11. Let N be the point in which the normal to a parabola at a point P, not the vertex, meets the axis. Prove that the projection on the axis of the line-segment PN is equal to one half the length of the latus rectum.

12. On a parabola, P is any point other than the vertex, and N is the point in which the normal at P meets the axis. Show that P and N are equally distant from the focus.

13. The tangent to a parabola at a point P, not the vertex, meets the directrix in the point L. Prove that the segment LP subtends a right angle at the focus.

14. Show that the length of a focal chord of the parabola $y^2 = 2mx$ is equal to $x_1 + x_2 + m$, where x_1, x_2 are the abscissæ of the end-points of the chord. Hence show that the mid-point of a focal chord is at the same distance from the directrix as it is from the end-points of the chord.

Exercises 15–26. The following exercises express properties of the parabola which involve an arbitrary point on the parabola. In order to prove these properties, it will, in general, be necessary to make actual use of the equation which expresses analytically the fact that the point lies on the parabola.

15. An arbitrary point P of a parabola, not the vertex, is joined with the vertex A, and a second line is drawn through P, perpendicular to AP, meeting the axis in Q. Prove that the projection on the axis of PQ is equal to the length of the latus rectum.

16. The tangent to a parabola at a point P, not the vertex, meets the tangent at the vertex in the point K. Show that the line joining K to the focus is perpendicular to the tangent at P.

17. The tangent to a parabola at a point P, not the vertex, meets the directrix and the latus rectum produced in points which are equally distant from the focus. Prove this theorem.

18. Prove that the coördinates of the point of intersection of the tangents to the parabola $y^2 = 2\,mx$ at the points (x_1, y_1), (x_2, y_2) may be put in the form

$$\left(\frac{y_1y_2}{2\,m},\ \frac{y_1 + y_2}{2}\right).$$

Suggestion. To reduce the coördinates to the desired form, use the equations which express analytically the fact that the two points lie on the parabola.

19. Show that the intercept on the axis of x of the line joining the points (x_1, y_1), (x_2, y_2) of the parabola $y^2 = 2\,mx$ may be expressed as

$$-\frac{y_1y_2}{2\,m}.$$

By means of the results of the two preceding exercises prove the following theorems.

20. The point of intersection of two tangents to a parabola and the point of intersection with the axis of the line joining their points of contact are equally distant from the tangent at the vertex, and are either on it or on opposite sides of it.

21. Tangents to a parabola at the end-points of a focal chord meet at right angles on the directrix.

22. If the points of contact of two tangents to a parabola are on the same side of the axis and at distances from the axis whose product is the square of half the length of the latus rectum, the tangents intersect on the latus rectum produced.

23. The end-points of a chord of a parabola, which subtends a right angle at the vertex, are on opposite sides of the axis and at distances from the axis, whose product is the square of the length of the latus rectum.

24. The chords of a parabola, which subtend a right angle at the vertex, pass through a common point on the axis; this point is at a distance from the vertex equal to the latus rectum.

25. The distance from the focus of a parabola to the point of intersection of two tangents is a mean proportional between the focal radii to the points of tangency.

26. The tangents to a parabola at the points P and Q intersect in T, and the normals at P and Q meet in N. Then the segment TM, where M is the mid-point of TN, subtends a right angle at the focus.

Locus Problems

27. Show that the locus of a point which moves so that the difference of the slopes of the lines joining it to two fixed points is constant is a parabola through the two fixed points What are its axis and vertex?

28. Determine the locus of a point which moves so that its distance from a fixed circle equals its distance from a fixed line passing through the center of the circle.

Ans. Two equal parabolas, with foci at the center of the circle and axes perpendicular to the fixed line.

29. The base of a triangle is fixed and the sum of the trigonometric tangents of the base angles is constant. Find the locus of the vertex.

CHAPTER VII

THE ELLIPSE

1. Definition. An ellipse is defined as the locus of a point P, the sum of whose distances from two given points, F and F', is constant. It is found convenient to denote this constant by $2a$. Then

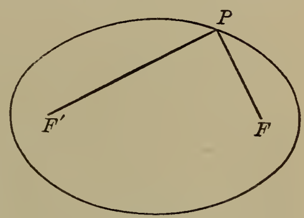

Fig. 1

(1) $$FP + F'P = 2a.$$

It is understood, of course, that P always lies in a fixed plane passing through F and F'.

The points F and F' are called the *foci* of the ellipse. It is clear that $2a$ must be greater than the distance between them.

Mechanical Construction. From the definition of the ellipse a simple mechanical construction readily presents itself. Let a string, of length $2a$, have its ends fastened at F and F', and let the string be kept taut by a pencil point at P. As the pencil moves, its point obviously traces out on the paper the ellipse.

The student will find it convenient to use two thumb tacks partially inserted at F and F'. A silk thread can be tied to one of the thumb tacks and wound round the other so that it will not slip. Thus a variety of ellipses with different foci and different values of a can be drawn.

Let the student make finally *one* ellipse in this manner, and draw it neatly.

Center, Vertices, Axes. It is obvious from the definition, — and the fact becomes more striking from the mechanical construction, — that the ellipse is symmetric in the line through the foci. It is also symmetric in the perpendicular bisector of FF'. Hence it is symmetric, furthermore, in the mid-point, O, of the line FF'.

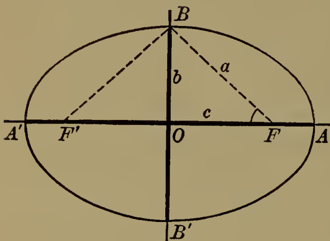

FIG. 2

The indefinite line through the foci, F and F', is called the *transverse axis* of the ellipse; the perpendicular bisector of FF', the *conjugate axis.* The point O is called the *center* of the ellipse; the points A, A', its *vertices.*

The line-segments AA' and BB', which measure the length and breadth of the ellipse, are known respectively as the *major axis* and the *minor axis* of the ellipse. The word "axes" refers sometimes to the transverse and conjugate axes, and sometimes to the major and minor axes, or their lengths, the context making clear in any case the meaning.

When P is at A, equation (1) becomes

$$FA + F'A = 2a.$$

But $$FA = A'F'.$$

Hence $$AA' = 2a \qquad \text{and} \qquad OA = a.$$

Thus it appears that the length of the *semi-axis major*, OA, is a. Let the length of the minor axis be denoted by $2b$, and the distance between the foci by $2c$. Then, from the triangle FOB, we have:

(2) $$a^2 = b^2 + c^2.$$

Note that, of the three quantities a, b, and c, the quantity a is always the largest.

Eccentricity. All circles have the same shape, *i.e.* are similar; and the same is true of parabolas. But it is not true of ellipses. As a measure of the roundness or flatness of an ellipse a number, called the *eccentricity*, has been chosen; this number is defined as the radio c/a and is denoted by e:

$$(3) \qquad e = \frac{c}{a}.$$

Since c is always less than a, it is seen that the eccentricity of an ellipse is always *less* than unity:

$$e < 1.$$

In terms of a and b, e has the value:

$$(4) \qquad e = \frac{\sqrt{a^2 - b^2}}{a}.$$

All ellipses with the same eccentricity are similar, and conversely. For the shape of an ellipse depends only on b/a, the ratio of its breadth to its length, and since from (4)

$$e = \sqrt{1 - \left(\frac{b}{a}\right)^2},$$

all ellipses for which the ratio b/a is the same have the same eccentricity, and conversely.

A circle is the limiting case of an ellipse whose foci approach each other, the length $2a$ remaining constant. The eccentricity approaches 0, and a circle is often spoken of as an ellipse of eccentricity 0.

EXERCISES

1. The semi-axes of an ellipse are of lengths 3 cm. and 5 cm. Find the distance between the foci, and the eccentricity.

Ans. 8; $\frac{4}{5}$.

2. The eccentricity of an ellipse is $\frac{3}{5}$ and the semi-axis minor is 4 in. long. How long is the major axis?

3. The major axis of an ellipse is twice as great as the minor axis. What is the eccentricity of the ellipse?

4. The major axis of an ellipse is 39 yards, and the eccentricity, $\frac{5}{13}$. Find the minor axis.

5. Express the eccentricity of an ellipse in terms of b and c.

6. Show, from Fig. 2, that the eccentricity is given by the formula

$$e = \cos OFB.$$

7. Give a proof, based on similar triangles, that two ellipses having the same eccentricity are similar.

2. Geometrical Construction. Points on the ellipse may be obtained with speed and accuracy by a simple geometrical construction. Draw the major axis and mark the points A, F, F', A' on it. Mark an arbitrary point Q between F and F'. With F as center and AQ as radius describe a circle, and with F' as center and $A'Q$ as radius describe a second circle.

Fig. 3

The points of intersection of these two circles will lie on the ellipse, since the sum of the radii is

$$AQ + A'Q = 2a.$$

It is, of course, not necessary to draw the complete circles, but only so much of them as to determine their points of intersection. Moreover, *four* points, instead of *two*, can be obtained from each pair of settings of the compasses by simply reversing the rôles of F and F'.

EXERCISES

1. Construct the ellipse for which $c = 2\frac{1}{2}$ cm., $a = 4$ cm.

2. From the ellipse just constructed make a templet, with holes at the foci and with the axes properly drawn.

3. Construct the ellipse whose axes are 4 cm. and 6 cm.

3. Equation of the Ellipse. It is natural to choose the axes of the ellipse as the coördinate axes (Fig. 4). Let the foci lie on the axis of x, and let P: (x, y) be any point of the ellipse. Then, from (1), § 1.

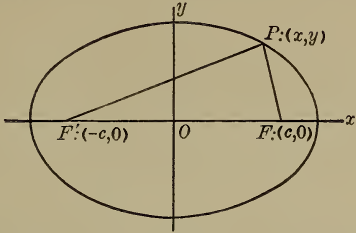

FIG. 4

(1) $$\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}=2a.$$

Transpose one of the radicals and square:

$$(x-c)^2+y^2=(x+c)^2+y^2-4a\sqrt{(x+c)^2+y^2}+4a^2.$$

Hence

(2) $$a\sqrt{(x+c)^2+y^2}=a^2+cx.$$

To remove this radical, square again:

(3) $$a^2x^2+2a^2cx+a^2c^2+a^2y^2=a^4+2a^2cx+c^2x^2,$$

or $$(a^2-c^2)x^2+a^2y^2=a^2(a^2-c^2).$$

But, by (2), § 1, $$a^2-c^2=b^2,$$

and hence

(4) $$b^2x^2+a^2y^2=a^2b^2,$$

or

(5) $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

This is the standard form of the equation of the ellipse, referred to its axes as the axes of coördinates. The proof, however, is not as yet complete, for it remains to show, conversely, that any point (x, y) whose coördinates satisfy equation (5) is a point of the ellipse. To do this, it is sufficient to show that x, y satisfy (1). From (5) we mount up to (4) and thence to (3), since all of these are equivalent equations. When, however, we extract a square root we obtain *two* equations each time, and so we are led, finally, to the *four* equations

$$\pm\sqrt{(x-c)^2+y^2}\pm\sqrt{(x+c)^2+y^2}=2a,$$

the ambiguous signs being chosen in all possible ways. The four equations can be characterized as follows:

i)	$+ \quad +$;	ii)	$+ \quad -$;
iii)	$- \quad +$;	iv)	$- \quad -$.

We wish to show that i) is the only possible one of the four equations. This is done as follows.

Equation iv) is satisfied by no pair of values for x and y, since the left-hand side is always negative and so can never be equal to the positive quantity $2a$.

Equations ii) and iii) say that the difference of the distances of (x, y) from F and F' is equal to $2a$, and hence greater than the line $FF'=2c$. Thus, in the triangle FPF' the difference of two sides is greater than the third side, and this is absurd.* Hence equations ii) and iii) are impossible and equation i) alone remains, q. e. d.

Consequently, if we start with equation (5) as given and require that $a>b$, then (5) represents an ellipse with semi-axes a and b and foci in the points $(\pm c, 0)$, where $c=\sqrt{a^2-b^2}$.

The Focal Radii. From equation (2) we obtain a simple expression for the length of the *focal radius*, $F'P$. Dividing (2) by a and remembering that $c/a=e$, we have:

$$\sqrt{(x+c)^2+y^2}=a+ex.$$

But the value of the left-hand side of this equation is precisely $F'P$. Hence

$$F'P=a+ex. \tag{6}$$

* If, in particular, the point (x, y) lay on FF', we should not, it is true, have a triangle. But it is at once obvious that in this case, too, equations ii) and iii) are impossible.

If, in transforming (1), the other radical had been transposed to the right-hand side and we had then proceeded as before, we should have found the equation:

$$a\sqrt{(x-c)^2+y^2}=a^2-cx.$$

From this we infer that

$$\sqrt{(x-c)^2+y^2}=a-ex,$$

or

(7) $$FP=a-ex.$$

EXERCISES

1. What is the equation of the ellipse whose axes are of lengths 6 cm. and 10 cm.? *Ans.* $\frac{x^2}{25}+\frac{y^2}{9}=1.$

2. Find the coördinates of the foci of the ellipse of Ex. 1.

3. The foci of an ellipse are at the points (1, 0) and (− 1, 0), and the minor axis is of length 2. Find the equation of the ellipse. *Ans.* $x^2+2y^2=2.$

4. Find the lengths of the axes, the coördinates of the foci, and the eccentricity of the ellipse

$$25x^2+169y^2=4225.$$

5. An ellipse, whose axes are of lengths 8 and 10, has its center at the origin and its foci on the axis of y. Obtain its equation.

6. Show that, if $B>A$, the equation

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}=1$$

still represents an ellipse with its axes lying along the axes of coördinates; but the foci lie on the axis of y at the points $(0, C)$ and $(0, -C)$, where

$$B^2=A^2+C^2.$$

The eccentricity is

$$e=\frac{C}{B}.$$

7. Find the lengths of the axes, the coördinates of the foci, and the value of the eccentricity for each of the following ellipses:

(*a*) $9x^2+4y^2=36$; (*d*) $5x^2+3y^2=45$;
(*b*) $3x^2+2y^2=12$; (*e*) $2x^2+7y^2=10$;
(*c*) $x^2+2y^2=4$; (*f*) $11x^2+y^2=3$.

4. Tangents. The ellipse has the remarkable property that the *tangent to the curve at any point makes equal angles with the focal radii drawn to that point:*

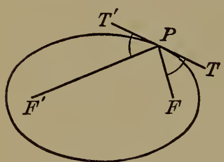

Fig. 5

$$\measuredangle FPT=\measuredangle F'PT'.$$

i) *Mechanical Proof.* The simplest proof of this theorem is a mechanical one. Think of a flexible, inelastic string of length $2a$ with its ends fastened at the foci, F and F'. Suppose a small, smooth bead to be threaded on this string. Let a cord be fastened to the bead and then pulled taut, so that the cord and the two portions of the string will be under tension. Evidently, the bead can be held in this manner at any point. (No force of gravity is supposed to act. The strings and bead may be thought of as resting on a smooth horizontal table.)

The forces that act on the bead are:

(*a*) the tension S in the cord;

(*b*) two equal tensions, R, in the string, directed respectively toward the foci.*

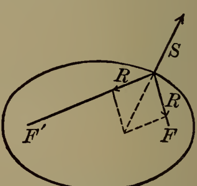

Fig. 6

Draw the parallelogram of forces for the forces R. It will be a rhombus, and so the resultant of these forces will bisect the angle between the focal radii.

On the other hand, the force S, equal and opposite to this resultant, is perpendicular to the tangent at P. In fact, if

* Since the bead is *smooth*, the tension in the string is the same at all its points, and so, in particular, is the same on the two sides of the bead.

instead of the flexible string we had a *smooth rigid wire*, in the form of the ellipse, for the bead to slide on, the bead would be held at P by the cord exactly as before. But the reaction of a smooth wire is at right angles to its tangent. This is the very conception of a *smooth* wire. For otherwise, if S were oblique, it could be resolved into a normal and a tangential component. But the smooth wire could not yield a reaction, part of which is along the tangent.

It follows, then, that the normal at P bisects the angle between the focal radii, and hence these make equal angles with the tangent at P, q. e. d.

ii) *Proof by Means of Minimum Distances. A Lemma.* A barnyard is bounded on one side by a straight river. The cows, as they come from the pasture, enter the barnyard by a gate at A, go to the river to drink, and then keep on to the door of the barn at B. What point, P, of the river should a cow select, in order to save her steps so far as possible?

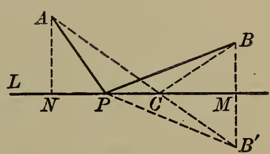

FIG. 7

It is easy to answer this question by means of a simple construction. From B drop a perpendicular BM on the line of the river bank, L, and produce it to B', making $MB' = BM$. Join A with B', and let AB' cut L at C. Then C is the position of P, for which the distance under consideration,

$$AP + PB,$$

is least.

For, the straight line AB' is shorter than any broken line APB':

$$AB' < APB'.$$

But $\qquad PB = PB' \qquad$ and $\qquad CB = CB'$.

Hence

$$AB' = AC + CB \qquad \text{and} \qquad APB' = AP + PB.$$

It follows, then, that

$$AC + CB < AP + PB,$$

if P is any point of L distinct from C. Hence C is the point for which APB is a minimum.

The point C is evidently characterized by the fact that

$$\measuredangle ACN = \measuredangle BCM.$$

We can state the result, then, by saying that *the point P, for which the distance APB is least, is the point for which*

$$\measuredangle APN = \measuredangle BPM.$$

Optical Interpretation. We have used a homely example of cows and a barnyard. The problem we have solved is, however, identical with the optical problem of finding the point at which a ray of light, emanating from A, will strike a plane mirror L, if the reflected ray is to pass through B. For, the law of light is, that it will travel the distance in the shortest possible time, and hence it will choose the shortest path.

Application to the Ellipse. The application of this result to the ellipse is as follows. The tangent to any smooth, closed, convex curve evidently is characterized by the fact that it meets the curve in one, and only one, point.

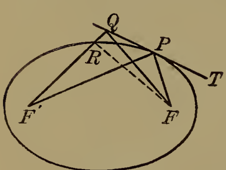

Fig. 8

Let P be any point of the ellipse. Draw the tangent, T, at P. Let Q be any point of T distinct from P. Now

$$F'P + FP = F'R + FR,$$

since the sum of the focal radii is the same for all points of an ellipse. But

$$FR < RQ + FQ$$

and so

$$F'R + FR < F'R + RQ + FQ = F'Q + FQ.$$

Therefore

$$F'P + FP < F'Q + FQ.$$

Hence P is that point of T for which the distance $F'QF$ is least, and consequently the lines $F'P$ and FP make equal angles with T, q.e.d.

EXERCISE

Show that the normal of an ellipse at any point distinct from the vertices A, A' cuts the major axis at a point which lies between the foci.

5. Optical and Acoustical Meaning of the Foci. Let a thin strip of metal, — say, a strip of brass a yard long and a quarter of an inch wide, — be bent into the form of an ellipse and polished on the concave side. Let a light be placed at one of the foci. Then the rays, after impinging on the metal, will be reflected and will come together again at the other focus, which will, therefore, be brilliantly illuminated.*

The same is true of heat, since heat rays are reflected from a polished surface by the same law as that of light rays. If, then, a candle is placed at one focus and some gunpowder at the other, the powder can be ignited by the heat from the candle.

Sound waves behave in a similar manner. The story is told of the Ratskeller in Bremen, the walls of which are shaped somewhat like an ellipse, that the city fathers were remarkably well informed concerning the feelings and views of the populace. For, the former drank their wine at a table which was situated at a focus, and thus could hear distinctly the conversation at a distant table, which stood at the other focus and about which the Bürger congregated.

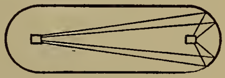

FIG. 9

6. Slope and Equation of the Tangent. The student will next turn to Ch. IX, § 2, where the slope of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

* The statement is, of course, strictly true only for such rays as travel in the plane through the foci, which is perpendicular to the elements of the cylinder formed by the polished band. Since, however, only a narrow strip of this cylinder is used, other rays will pass very near to the second focus and contribute to the illumination there.

at the point (x_1, y_1) is found to be

$$\lambda = -\frac{b^2 x_1}{a^2 y_1}. \tag{1}$$

The equation of the tangent line at this point is

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1. \tag{2}$$

Latus Rectum. The latus rectum of an ellipse is defined as a chord perpendicular to the major axis and passing through a focus. The term is also used to mean the *length* of such a chord.

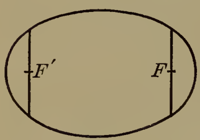

FIG. 10

Thus, in the ellipse

$$\frac{x^2}{25} + \frac{y^2}{16} = 1,$$

one focus is at the point (3, 0). The length of the latus rectum is twice that of the positive ordinate corresponding to this point. Setting, then, $x = 3$ in the equation of the curve and solving for that ordinate, we have

$$\frac{y^2}{16} = 1 - \frac{9}{25} = \frac{16}{25}, \qquad y = \frac{16}{5} = 3\tfrac{1}{5}.$$

Hence the length of the latus rectum is $6\frac{2}{5}$.

EXERCISES

1. Find the equation of the tangent to the ellipse

$$\frac{x^2}{225} + \frac{y^2}{25} = 1$$

at the point (9, 4). *Ans.* $x + 4y = 25$.

2. Find the equation of the normal to the ellipse of Ex. 1 at the same point. *Ans.* $4x - y = 32$.

3. At what point does the tangent to the ellipse

$$2x^2 + 3y^2 = 14$$

at the point $(-1, 2)$ cut the axis of y?

4. At what angle does the straight line through the origin, which bisects the angle between the positive axes of coördinates, cut the ellipse $3x^2 + 4y^2 = 7$? *Ans.* 81° 53′.

5. Find the area of the triangle cut off from the first quadrant by the tangent to the ellipse of Ex. 3 at the point (1, 2). *Ans.* $8\frac{1}{6}$.

6. Find the length of the latus rectum of the ellipse of Ex. 1. *Ans.* $3\frac{1}{3}$.

7. The same for the ellipse of Ex. 3.

8. Show that the length of the latus rectum of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad b < a,$$

is given by any one of the expressions

$$\frac{2b^2}{a}; \qquad 2b\sqrt{1-e^2}; \qquad 2a(1-e^2).$$

Find its value in terms of c and e.

9. Find the length of the latus rectum of the ellipse

$$25x^2 + 16y^2 = 400. \qquad \textit{Ans. } 6\tfrac{2}{5}.$$

10. Prove that the minor axis of an ellipse is a mean proportional between the major axis and the latus rectum.

7. A New Locus Problem. Given a line D and a point F distant m from D. To find the locus of a point P such that the ratio of its distance FP from F to its distance MP from D is always equal to a given number, ϵ:

(1) $$\frac{FP}{MP} = \epsilon, \qquad \text{or} \qquad FP = \epsilon MP.$$

It is understood that P shall be restricted to the plane determined by F and D.

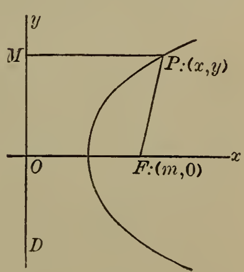

FIG. 11

If, in particular, $\epsilon = 1$, the locus is a parabola with D as directrix and F as focus; Ch. VI, § 1.

To treat the general case, let D be taken as the axis of y and let the positive axis of x pass through F. Then

$$FP = \sqrt{(x-m)^2 + y^2}, \qquad MP = \pm x,$$

the lower sign holding only when x is negative, and (1) becomes

$$(2) \qquad \sqrt{(x-m)^2 + y^2} = \pm \epsilon x.$$

On squaring and transposing we obtain the equation:

$$(3) \qquad (1-\epsilon^2)x^2 - 2mx + y^2 + m^2 = 0.$$

This is the equation of the proposed locus.

The student will now turn to Ch. XI and study carefully § 1.

EXERCISES

1. Take $\epsilon = \frac{1}{2}$ and $m = 3$, the unit of length being 1 cm. With ruler and compasses construct a generous number of points of the locus,* and then draw in the locus with a clean, firm line.

2. Work out the equation of the locus of Ex. 1 directly, using the method of the foregoing text, but not looking at the formulas. *Ans.* $3x^2 + 4y^2 - 24x + 36 = 0$.

3. Take $\epsilon = \frac{3}{5}$ and $m = 4$, the unit of length being 1 cm. Draw the locus accurately, as in Ex. 1.

4. Work out directly the equation of the locus of Ex. 3. *Ans.* $16x^2 + 25y^2 - 200x = -400$.

5. By means of a transformation to parallel axes show that the curve of Ex. 2 is an ellipse whose center is at the point (4, 0) and whose axes are of lengths 4 and $2\sqrt{3}$. What is its eccentricity?

* The details of the construction are an obvious modification of the corresponding construction for the parabola in Ch. VI, § 1. A circle of arbitrary radius is drawn with its center at F, and this circle is cut by a parallel to D, whose distance from D is *twice* the radius of the circle.

6. Show that the curve of Ex. 4 is an ellipse whose axes are $7\frac{1}{2}$ and 6. What is its eccentricity?

8. Discussion of the Case $\epsilon < 1$. The Directrices. From equation (3) of § 7 follows:

$$(1)\qquad x^2 - \frac{2m}{1-\epsilon^2}x + \frac{y^2}{1-\epsilon^2} = -\frac{m^2}{1-\epsilon^2}.$$

The first two terms on the left-hand side are also the first two in the expansion of

$$\left(x - \frac{m}{1-\epsilon^2}\right)^2 = x^2 - \frac{2m}{1-\epsilon^2}x + \frac{m^2}{(1-\epsilon^2)^2}.$$

If, then, we add the third term of the last expression to both sides of (1), we shall have:

$$x^2 - \frac{2m}{1-\epsilon^2}x + \frac{m^2}{(1-\epsilon^2)^2} + \frac{y^2}{1-\epsilon^2} = \frac{m^2}{(1-\epsilon^2)^2} - \frac{m^2}{1-\epsilon^2},$$

or

$$(2)\qquad \left(x - \frac{m}{1-\epsilon^2}\right)^2 + \frac{y^2}{1-\epsilon^2} = \frac{\epsilon^2 m^2}{(1-\epsilon^2)^2}.$$

This equation reminds us strongly of the equation of an ellipse. In fact, if we transform to parallel axes with the new origin, O', at the point

$$x_0 = \frac{m}{1-\epsilon^2}, \qquad y_0 = 0,$$

the equations of transformation are

$$(3)\qquad x' = x - \frac{m}{1-\epsilon^2}, \qquad y' = y,$$

and (2) then takes on the form

$$(4)\qquad x'^2 + \frac{y'^2}{1-\epsilon^2} = \frac{\epsilon^2 m^2}{(1-\epsilon^2)^2},$$

or

$$(5)\qquad \frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1,$$

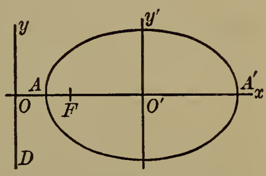

FIG. 12

where

$$(6)\qquad a = \frac{\epsilon m}{1-\epsilon^2}, \qquad b = \frac{\epsilon m}{\sqrt{1-\epsilon^2}}.$$

Thus the locus is seen to be an ellipse with its center, O', at the point

(7) $$\left(\frac{m}{1-\epsilon^2}, 0\right),$$

the semi-axes being given by (6).

The value of c is given by the equation $c^2 = a^2 - b^2$. Hence

(8) $$c = \frac{\epsilon^2 m}{1-\epsilon^2}.$$

The eccentricity, $e = c/a$, is now seen to be precisely ϵ:

$$e = \epsilon;$$

i.e. *the given constant,* ϵ, *turns out to be the eccentricity of the ellipse.*

Finally, *F is one of the foci.* For, the distance from F to O' is

$$OO' - OF = \frac{m}{1-\epsilon^2} - m = \frac{\epsilon^2 m}{1-\epsilon^2},$$

and this, by (8), is precisely c.

The line D is called a *directrix* of the ellipse. Its distance from the center is

$$OO' = \frac{m}{1-\epsilon^2} = \frac{m\epsilon}{1-\epsilon^2}\frac{1}{\epsilon} = \frac{a}{\epsilon}.$$

The Directrices. From the symmetry of the ellipse it is clear that there is a second directrix, D', on the other side of the conjugate axis, parallel to that axis, and at the same distance from it as D. This line D' and the focus F' stand in the same relation to the ellipse as the first line, D, and the focus F. Thus the ellipse is the locus of a point so moving that its distance from a focus always bears to its distance from the corresponding directrix the same ratio, e, the eccentricity.

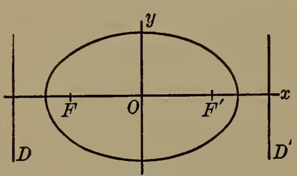

Fig. 13

Since the distance of D from the center of the ellipse is a/e, the equations of the directrices of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad a>b,$$

are

$$x=-\frac{a}{e}, \qquad x=\frac{a}{e}.$$

EXERCISES

1. Show that the distances of the vertices, A and A', from O are:

$$OA=\frac{m}{1+e}, \qquad OA'=\frac{m}{1-e}.$$

2. Collect the foregoing results in a syllabus, arranged in tabular form, giving each of the quantities a, b, c, OO', OA, OA', OF, OF' in terms of m and e.

3. Work out each of the quantities of Ex. 2 directly for the ellipse of § 7, Ex. 4, and verify the result by substituting the values $e=\frac{3}{5}$, $m=4$ in the formulas of the syllabus.

4. Between the *five* constants of the ellipse, a, b, c, e, m, there exist *three* relations, which may be written in a variety of ways; as, for example,

$$\text{i) } a^2=b^2+c^2; \qquad \text{ii) } e=\frac{c}{a}; \qquad \text{iii) } m=\frac{1-e^2}{e}a.$$

By means of these relations, any three of the five quantities can be expressed in terms of the other two. Thus, in Ex. 2, m and e are chosen as the quantities in terms of which all others shall be expressed.

Taking the semi-axes, a and b, $(a>b)$, as the preferred pair, express the other quantities in terms of them.

5. Show that the tangent to the ellipse

$$\frac{x^2}{25}+\frac{y^2}{16}=1$$

at an extremity of a latus rectum cuts the transverse axis in the same point in which this axis is cut by a directrix.

6. The same for any ellipse.

7. Prove directly that, if P is any point of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad b<a,$$

the ratio of its distance from a focus to its distance from the corresponding directrix is equal to the eccentricity.

8. Show that in an ellipse the major axis is a mean proportional between the distance between the foci and the distance between the directrices.

9. Show that the distances from the center and a focus of an ellipse to the directrix corresponding to the focus are in the same ratio as the squares of the semi-axis major and the semi-axis minor.

9. The Parabola as the Limit of Ellipses. We have proved that, when $\epsilon<1$, equation (3), § 7, represents an ellipse with eccentricity $e=\epsilon$. We know that, if $\epsilon=1$, the equation represents a parabola. If, then, in the equation we allow ϵ to approach 1 through values <1, the ellipse which the equation defines approaches a parabola as its limit.

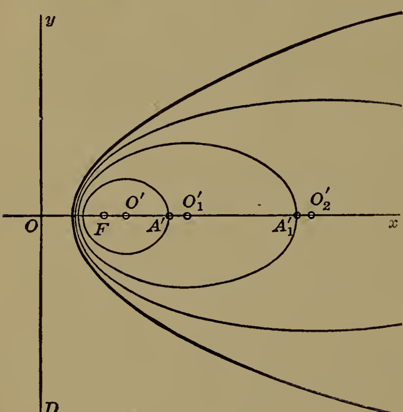

FIG. 14

We can visualize the ellipse, going over into a parabola, by drawing a number of ellipses having the same value of m, but having values for ϵ which are increasing toward 1 as their limit, viz. $\epsilon=\frac{1}{2}$, $\epsilon=\frac{3}{4}$, $\epsilon=\frac{7}{8}$, $\cdots$. The directrix D, along the axis of y, and the focus $F:(m,\,0)$ are the same for all the ellipses. But the center O' and the right-hand vertex

A' of each successive ellipse are farther away from O, and their distances from O, namely,

$$OO' = \frac{m}{1-\epsilon^2}, \qquad OA' = \frac{m}{1-\epsilon},$$

increase without limit. Thus, as ϵ approaches 1, the ellipse approaches as its limit the parabola whose directrix is D and whose focus is F.

10. New Geometrical Construction for the Ellipse. Parametric Representation. Let it be required to draw an ellipse when its axes, AA' and BB', are given. Describe circles of radii $a = OA$ and $b = OB$, with the origin O as the common center. Draw any ray from O, making an angle ϕ with the positive axis of x, as shown in the figure. Through the points Q and R draw the parallels indicated. Their point of intersection, P, will lie on the ellipse. For, if the coördinates of P be denoted by (x, y), it is clear that

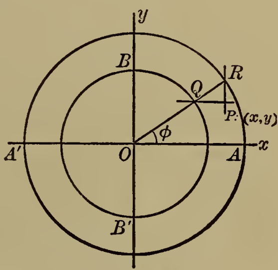

FIG. 15

(1) $$x = a\cos\phi, \qquad y = b\sin\phi.$$

From these equations ϕ can be eliminated by means of the trigonometric identity

$$\sin^2\phi + \cos^2\phi = 1.$$

Hence

(2) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Conversely, any point (x, y) on the ellipse (2) has corresponding to it an angle ϕ, for which equations (1) are true.

Equations (1) afford what is known as a *parametric representation* of the coördinates of a variable point (x, y) of the ellipse in terms of the *parameter* ϕ. When $b = a$, the ellipse becomes a circle, and the equations (1) become

(3) $$x = a\cos\phi, \qquad y = a\sin\phi.$$

These parametric representations, though little used in Analytic Geometry, are an important aid in the Calculus.

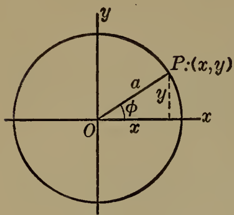

FIG. 16

The larger of the two circles in Fig. 15 is commonly called the *auxiliary circle* of the ellipse, and the points R and P are known as *corresponding points.* The angle ϕ is called the *eccentric* angle.

EXERCISE

By means of the foregoing method, draw on squared paper an ellipse whose axes are of length 4 cm. and 6 cm.

EXERCISES ON CHAPTER VII

1. The earth moves about the sun in an elliptic orbit.* The shortest and longest distances from it to the sun are in the ratio 29 : 30. What is the eccentricity of the orbit?

2. Show that the slopes of the tangents to an ellipse at the extremities of the latera recta are $\pm e$.

3. The axes of an ellipse which goes through the points (4, 1), (2, 2) are the axes of coördinates. Find its equation.

4. The center of an ellipse is in the origin and the foci are on the axis of x. The ellipse has an eccentricity of $\frac{3}{5}$ and goes through the point (12, 4). What is its equation?

$$\textit{Ans.}\quad \frac{x^2}{25}+\frac{y^2}{16}=\frac{169}{25}.$$

5. Solve the preceding problem if the foci may lie on either axis of coördinates.

6. Find the equations of the ellipses which have the axes of coördinates as axes, go through the point (3, 4), and have their major and minor axes in the ratio 3 : 2.

7. Show that the ellipses represented by the equation

$$2x^2+3y^2=c^2,$$

* The planets describe ellipses about the sun as a focus, and the comets usually describe parabolas with the sun as the focus.

where c^2 is an arbitrary positive constant, are similar. What is the common value of the eccentricity?

8. How many ellipses are there with eccentricity $\frac{1}{2}$, having their centers in the origin and their foci on the axis of x? Deduce an equation which represents them all.

Ans. $3x^2 + 4y^2 = c^2$.

9. The foci of an ellipse lie midway between the center and the vertices. What is the eccentricity? How many such ellipses are there, with centers in the origin and foci on the axis of x? Write an equation which represents them all.

10. The line joining the left-hand vertex of an ellipse with the upper extremity of the minor axis is parallel to the line joining the center with the upper extremity of the right-hand latus rectum. Answer the questions of the preceding exercise.

11. The foci of an ellipse subtend a right angle at either extremity of the minor axis. What is the eccentricity? Find the equation of all such ellipses with centers in the origin and foci on the axis of y.

12. Prove that the ratio of the distance from a focus of an ellipse to the intersection with the transverse axis of the normal at a point P, and the distance from this focus to P equals the eccentricity of the ellipse.

13. The projections of a point P of an ellipse on the transverse and conjugate axes are P_1 and P_2. The tangent at P meets these axes in T_1 and T_2. Prove that $OP_1 \cdot OT_1 = a^2$ and $OP_2 \cdot OT_2 = b^2$, where O is the center and a and b are the semi-axes of the ellipse.

14. Prove that the segment of a tangent to an ellipse between the point of contact and a directrix subtends a right angle at the corresponding focus.

15. Determine the points of an ellipse at which the tangents have intercepts on the axes whose absolute values are proportional to the lengths of the axes.

16. Through a point M of the major axis of an ellipse a line is drawn parallel to the conjugate axis, meeting the ellipse in P and the tangent at an extremity of the latus rectum in Q. Show that the distance MQ equals the distance of P from the focus corresponding to the latus rectum taken.

17. Prove that the line joining a point P of an ellipse with the center and the line through a focus perpendicular to the tangent at P meet on a directrix.

18. Prove that the distance from a focus F to a point P of an ellipse equals the distance from F to the tangent to the auxiliary circle at the point corresponding to P.

19. Find the equation of a circle which is tangent to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

at both ends of a latus rectum.

20. In an ellipse whose major axis is twice the minor axis, a line of length equal to the minor axis has one end on the ellipse, the other on the conjugate axis. The two ends are always on opposite sides of the transverse axis. Prove that the mid-point of the line lies always on the transverse axis.

21. A number of ellipses have the same major axis both in length and position. A tangent is drawn to each ellipse at the upper extremity of the right-hand latus rectum. Prove that these tangents all pass through a point.

Exercises 22–28. In these exercises, in which properties involving an arbitrary point P of an ellipse are to be proved, it will, in general, be necessary to make actual use of the equation expressing the fact that the point P lies on the ellipse.

22. The tangent to an ellipse at a point P meets the tangent at one vertex in Q. Prove that the line joining the other vertex to P is parallel to the line joining the center to Q.

23. The lines joining the extremities of the minor axis with a point P of an ellipse meet the transverse axis in the points

M and N. Prove that the semi-axis major is a mean proportional between the distances from the center to M and N.

24. Prove the theorem of the preceding exercise when the major and minor axes, and the transverse and conjugate axes, are interchanged.

25. Show that the segment of a directrix, between the points of intersection of the lines joining the vertices with a point on an ellipse, subtends a right angle at the corresponding focus.

26. Prove that the product of the distances of the foci of an ellipse from a tangent is a constant, independent of the choice of the tangent.

27. Let F' and F be the foci of an ellipse and P any point on it. Prove that $b^2 : FK^2 = F'P : FP$, where FK is the distance from F to the tangent at P.

28. The normal to an ellipse at a point P meets the axes in N_1 and N_2. Show that $PN_1 \cdot PN_2$ is equal to the product of the focal radii to P.

Loci

29. A point moves so that the product of the slopes of the two lines joining it to two fixed points is a negative constant. What is its locus?

30. A circle whose diameter is 10 cm. is drawn, center at O. On a radius OA a point B is marked distant 4 cm. from O. If OQ is any second radius, show how to construct, with ruler and compasses, a point P on OQ, whose distance from the circle equals its distance from B. In this way plot a number of points on the locus of P.

31. Find the equation of the locus of the point P of the preceding exercise. Take the origin of coördinates at the mid-point of OB.

32. The base of a triangle is fixed and the product of the tangents of the base angles is a positive constant. Find the locus of the vertex.

CHAPTER VIII

THE HYPERBOLA

1. Definition. A *hyperbola* is defined as the locus of a point P, the difference of whose distances from two given points, F and F', is constant. It is found convenient to denote this constant by $2a$. Then

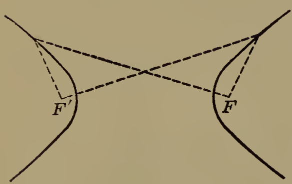

FIG. 1

$$FP - F'P = 2a,$$

or $$F'P - FP = 2a.$$

It is understood, of course, that P is restricted to a particular plane through F and F'.

The points F and F' are called the *foci* of the hyperbola. It is clear that $2a$ must be *less* than the distance between them. Denote this distance by $2c$.

Geometrical Construction. Draw the indefinite line FF', mark the mid-point, O, of the segment FF', and the points A and A' each at a distance a from O:

$$OA = OA' = a; \qquad OF = OF' = c.$$

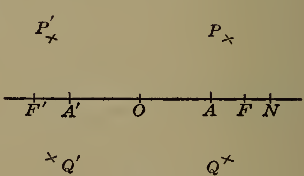

FIG. 2

The point A lies on the locus; for,

$$FA = c - a, \qquad F'A = c + a,$$

and hence $$F'A - FA = 2a.$$

Likewise, A' lies on the curve.

Mark any point, N, to the right of F. With radius AN and center F, describe a circle. Next, with radius $A'N$ and center

F', describe a second circle. The points P and Q in which these circles intersect are points of the locus. For,

$$F'P - FP = A'N - AN = A'A = 2a.$$

Two more points, P' and Q', can be obtained from the same pair of settings by interchanging the centers, F and F', of the circles.

By repeating the construction a number of times, a goodly array of points of the hyperbola can be obtained. These points will lie on two distinct arcs, symmetric to each other in the perpendicular bisector BOB' of FF'. Thus it will be seen that the hyperbola consists of two parts, or *branches*, as they are called. These branches, besides being the images of each other in BB', are each the image of itself in FF'. It is natural to speak of the indefinite straight lines FF' and BB' as the *axes* of the hyperbola. FF' is called the *transverse*, BB' the *conjugate* axis; O is the *center*, and A, A' are the *vertices*.

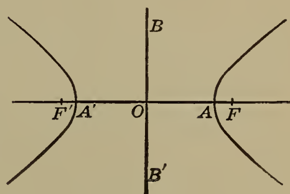

FIG. 3

EXERCISES

1. Taking $c = 3$ cm. and $a = 2$ cm., make a clean drawing of the corresponding hyperbola.

2. Reproduce the drawing on a rectangular card and, with a sharp knife or a small pair of scissors, cut out the center of the card along the hyperbola and two parallels to the transverse axis. On the templet which remains make holes at the foci and draw the two axes.

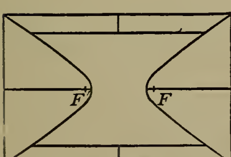

FIG. 4

2. Equation of the Hyperbola. The treatment here is parallel to that of the ellipse, Ch. VII, § 3. Let the transverse axis be chosen as the axis of x; the conjugate axis, as the axis of y. Then the equation of the right-hand branch of the hyperbola can be written in the form

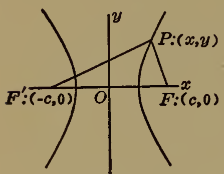

FIG. 5

$$(1) \quad \sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=2a.$$

Transpose the first radical and square:

$$(x-c)^2+y^2=(x+c)^2+y^2$$
$$-4a\sqrt{(x+c)^2+y^2}+4a^2.$$

Hence

$$(2) \qquad a\sqrt{(x+c)^2+y^2}=a^2+cx.$$

Square again:

$$a^2x^2+2a^2cx+a^2c^2+a^2y^2=a^4+2a^2cx+c^2x^2,$$

or

$$(3) \qquad (a^2-c^2)x^2+a^2y^2=a^2(a^2-c^2).$$

This is precisely the same equation that presented itself in the case of the ellipse; but the locus is a curve of wholly different nature. The reason is, that a and c have different *relative values*. In the ellipse, a was *greater* than c, and hence a^2-c^2 was positive. It could be denoted by b^2. Here, a is *less* than c; a^2-c^2 is negative, and it cannot be set equal to b^2. It can, however, be set equal to $-b^2$. This we will do:

$$(4) \qquad a^2-c^2=-b^2, \qquad \text{or} \qquad c^2=a^2+b^2,$$

thus *defining* the quantity b in the case of the hyperbola by the equation:

$$b=\sqrt{c^2-a^2}.$$

The final equation between x and y can now be written in the form

$$(5) \qquad \frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

This equation is satisfied by the coördinates of all points on the right-hand branch, as is seen from the way in which it was deduced. It is, however, also satisfied by the coördinates of all points on the left-hand branch. For such a point, the signs of both radicals in (1) will be reversed. Starting, now, with the new equation and proceeding as before, we find the same equation (3), which we may again write in the form (5), and thus the truth of the statement is established.

Is (5) satisfied by the coördinates of still other points? To answer this question, let (x, y) be *any* point whose coördinates satisfy (5). Then, starting from (5), we retrace our steps, admitting, each time that we extract a square root, both signs of the radical as conceivably possible. Thus we can be sure that (x, y) will satisfy *one* of the four equations

$$\pm\sqrt{(x+c)^2+y^2}\pm\sqrt{(x-c)^2+y^2}=2a,$$

corresponding to the four conceivable choices of the signs of the radicals:

i)	$-$	$+$;	iii)	$-$	$-$;
ii)	$+$	$-$;	iv)	$+$	$+$.

If (x, y) satisfies i) or ii), the point lies on the hyperbola. The other two cases are impossible. For, case iii) says that a negative quantity is equal to a positive quantity, and case iv) says that $F'P + FP = 2a$. Now $F'P + FP$, being the sum of two sides of the triangle FPF', is *greater* than the third side, FF', or $2c$. But $2a$ is actually *less* than $2c$. Hence we have a contradiction, and this case cannot arise.

We have shown then, finally, that (5) is the equation of the hyperbola.

EXERCISE

Plot the hyperbola

$$\frac{x^2}{25}-\frac{y^2}{16}=1$$

directly from its equation, taking 1 cm. as the unit of length.

3. Axes, Eccentricity, Focal Radii. The *transverse* and the *conjugate axis* have already been defined in § 1. The segment AA' of the transverse axis is called the *major axis*, and this term is also applied to its length, $2a$. The segment BB' of the conjugate axis, whose center is at O and whose length is

$2b$, is called the *minor axis*, and this term is also applied to its length, $2b$.

The major axis of an ellipse is always longer than the minor axis. In the case of the hyperbola, however, this is not always true. For example, if $2c$ and $2a$ are taken as 10 and 6 respectively, then $2b = 8$. Thus the major axis of the hyperbola is to be understood as the principal axis, but not necessarily as the longer axis.

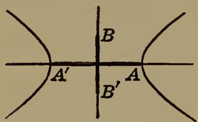

FIG. 6

The *eccentricity* of the hyperbola is defined as the number

$$e = \frac{c}{a}.$$

Since c is greater than a, the eccentricity of a hyperbola is always *greater* than unity.

The eccentricity characterizes the *shape* of the hyperbola. All hyperbolas having the same eccentricity are *similar*, differing only in the scale to which they are drawn, and conversely; cf. Exercise 8.

The *focal radii* FP, $F'P$ can be represented by simple expressions, similar to those which presented themselves in the case of the ellipse. On dividing equation (2), § 2, through by a, we have:

$$\sqrt{(x+c)^2 + y^2} = a + ex.$$

Hence, when P is a point of the right-hand branch,

(1) $$F'P = ex + a.$$

The evaluation,

(2) $$FP = ex - a,$$

is obtained in a similar manner.*

If P is a point of the left-hand branch, these formulas become:

(3) $$F'P = -(ex + a); \qquad FP = -(ex - a).$$

* P being a point of the right-hand branch, x is positive and greater than or equal to a; also, $e > 1$. Hence $ex > a$, and $ex - a$ is positive, as it should be.

EXERCISES

1. Find the lengths of the axes, the coördinates of the foci, and the value of the eccentricity for each of the following hyperbolas.

(a) $\frac{x^2}{16} - \frac{y^2}{9} = 1.$ *Ans.* 8, 6; (5, 0), (−5, 0); $1\frac{1}{4}$

(b) $x^2 - y^2 = a^2.$ *Ans.* $2a, 2a$; $(a\sqrt{2}, 0), (-a\sqrt{2}, 0)$; $\sqrt{2}$.

(c) $4x^2 - 3y^2 = 24.$

(d) $2x^2 - y^2 = 4.$

(e) $5x^2 - 6y^2 = 8.$

(f) $6x^2 - 9y^2 = 4.$

2. If the eccentricity of a hyperbola is 2 and its major axis is 3, what is the length of its minor axis? *Ans.* $3\sqrt{3}$.

3. How far apart are the foci of the hyperbola in Ex. 2? *Ans.* 6.

4. What is the equation of the hyperbola whose eccentricity is $\sqrt{2}$ and whose foci are distant 4 from each other? *Ans.* $x^2 - y^2 = 2$.

5. The extremities of the minor axis of a hyperbola are in the points (0, ± 3) and the eccentricity is 2. Find the equation of the hyperbola.

6. Show that, in terms of a and b, e has the value

$$e = \frac{\sqrt{a^2 + b^2}}{a}.$$

7. Express b in terms of a and e.

8. Prove that two hyperbolas which have the same eccentricity are similar, and conversely.

9. Establish formulas (3).

4. The Asymptotes. Two lines, called the *asymptotes*, stand in a peculiar and important relation to the hyperbola. They are the lines

$$y = \frac{bx}{a} \quad \text{and} \quad y = -\frac{bx}{a}.$$

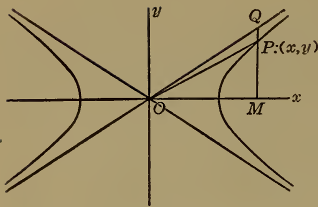

FIG. 7

Let a point $P:(x, y)$ move off along a branch of the hyperbola

(1) $$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

and let this take place, for definiteness, in the first quadrant. The slope of the line OP is

$$\frac{MP}{OM}=\frac{y}{x}.$$

Since the coördinates (x, y) of P satisfy (1), it follows that

(2) $$y=\frac{b}{a}\sqrt{x^2-a^2},$$

and hence

(3) $$\frac{y}{x}=\frac{b}{a}\sqrt{1-\frac{a^2}{x^2}}.$$

When P recedes indefinitely, x increases without limit, and the right-hand side of this equation approaches the limit b/a. Thus we see that the slope of OP approaches that of the line OQ,

(4) $$y=\frac{b}{a}x,$$

as its limit, always remaining, however, less than the latter slope, so that P is always below OQ.

It seems likely that P will come indefinitely near to this line; but this fact does not follow from the foregoing, since P might approach a line parallel to (4) and lying below it. In that case, all that has been said would still be true.

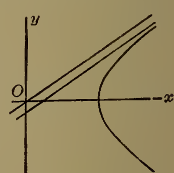

FIG. 8

That P does, however, actually approach (4) can be shown by proving that the distance PQ approaches 0 as its limit. Now,

$$PQ=MQ-MP,$$

and, from (4),

$$MQ=\frac{b}{a}x.$$

Furthermore, MP is the y-coördinate of the point P on the hyperbola:

$$MP = \frac{b}{a}\sqrt{x^2 - a^2}.$$

Hence $$PQ = \frac{b}{a}[x - \sqrt{x^2 - a^2}].$$

To find the limit approached by the square bracket, we resort to an algebraic device. The value of the bracket will clearly not be changed if we multiply and divide it by the expression $x + \sqrt{x^2 - a^2}$:

$$x - \sqrt{x^2 - a^2} = \frac{(x - \sqrt{x^2 - a^2})(x + \sqrt{x^2 - a^2})}{x + \sqrt{x^2 - a^2}}.$$

But the numerator of the last expression reduces at once to a^2. Hence

$$x - \sqrt{x^2 - a^2} = \frac{a^2}{x + \sqrt{x^2 - a^2}}.$$

From this form it is evident that the bracket approaches 0 when x increases indefinitely; and hence the limit of PQ is zero,* q. e. d.

Similar reasoning, or considerations of symmetry, applied in the other quadrants, show that in the second and fourth quadrants P approaches the line

(5) $$y = -\frac{b}{a}x,$$

while in the third quadrant, as in the first, P approaches (4).

The equations (4) and (5), of the asymptotes, can also be written in the form

$$\frac{x}{a} - \frac{y}{b} = 0, \qquad \frac{x}{a} + \frac{y}{b} = 0.$$

* The limit approached by the variable $x - \sqrt{x^2 - a^2}$ can be found geometrically as follows. Construct a variable right triangle, one leg of which is fixed and of length a, the hypothenuse being variable and of length x. Then the above variable, $x - \sqrt{x^2 - a^2}$, is equal to the difference in length between the hypothenuse and the variable leg. This difference obviously approaches 0 as x increases indefinitely.

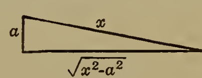

FIG. 9

It is easy to remember these equations, since they can be written down by replacing the right-hand side of (1) by 0, factoring the left-hand side:

$$\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{a}+\frac{y}{b}\right)=0,$$

and putting the individual factors equal to zero.

The slopes of the asymptotes are b/a and $-b/a$. Consequently, the asymptotes make equal angles with the transverse axis.

Since the ratio of b to a is unrestricted, the asymptotes can make any arbitrarily assigned angle with each other. If, in particular, $b=a$, this angle is a right angle, and the curve is called a *rectangular*, or *equilateral, hyperbola.* Its equation can be written in the form:

(6) $$x^2-y^2=a^2.$$

Its eccentricity is $e=\sqrt{2}$.

Construction of the Asymptotes. Mark with heavy lines the major and minor axes, and through the extremities of each draw lines parallel to the other, thus obtaining a rectangle. The diagonals of this rectangle, produced, are the asymptotes, since their slopes are clearly $\pm b/a$.

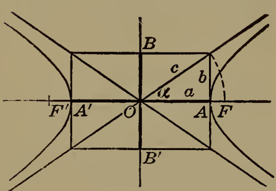

FIG. 10

The diagonals of the rectangle have lengths equal to the distance $2c$ between the foci, for, $c^2=a^2+b^2$ and the lengths of the sides of the rectangle are $2a$ and $2b$. If the acute angle between an asymptote and the transverse axis is denoted by α, then

$$e=\sec\alpha.$$

EXERCISES

1. Find the equations and slopes of the asymptotes of the hyperbolas of Exercise 1, § 3. Draw the hyperbolas.

2. Show that the asymptotes of the hyperbola

$$Ax^2 - By^2 = C,$$

where A, B, and C are any three positive quantities, are given by the equations

$$\sqrt{A}x + \sqrt{B}y = 0, \qquad \sqrt{A}x - \sqrt{B}y = 0.$$

3. Find the equation of the hyperbola whose asymptotes make angles of 60° with the axis of x and whose vertices are situated at the points (1, 0), and (– 1, 0). *Ans.* $3x^2 - y^2 = 3$.

4. Show that the slopes of the asymptotes are given by the expression $\pm\sqrt{e^2 - 1}$.

5. The slope of one asymptote of a hyperbola is $\frac{3}{4}$. Find the eccentricity. *Ans.* $e = 1\frac{1}{4}$.

6. The distance of a focus of a certain hyperbola from the center is 10 cm., and the distance of a vertex from the focus is 2 cm. What angle do the asymptotes make with the conjugate axis? *Ans.* 53° 8′.

7. Show that the circle circumscribed about the rectangle of the text passes through the foci.

8. A perpendicular dropped from a focus F on an asymptote meets the latter at E. Show that $OE = a$, and $EF = b$.

9. Find the equation of the equilateral hyperbola whose foci are at unit distance from the center.

10. Find the equation of the equilateral hyperbola which passes through the point (– 5, 4).

5. **Tangents.** The method of finding the slope of an ellipse, Ch. IX, § 2, can be applied to the hyperbola, and it is thus shown that the slope of this curve,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

at the point (x_1, y_1) is

$$\lambda = \frac{b^2x_1}{a^2y_1}.$$

The equation of the tangent of the hyperbola at this point is

(1) $$\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1.$$

THEOREM. *The tangent of a hyperbola at any point bisects the angle between the focal radii.*

To prove this proposition we recall the theorem of Plane Geometry which says that the bisector of an angle of a triangle divides the opposite side into segments which are proportional to the adjacent sides. It is easily seen that the converse * of this proposition is also true, and hence it is sufficient for our proof to show that

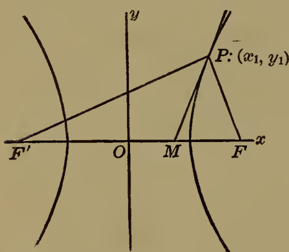

FIG.

(2) $$\frac{FP}{FM} = \frac{F'P}{F'M}.$$

We already have simple expressions for the numerators. If P: (x_1, y_1) be a point of the right-hand branch of the curve, then, by § 3,

$$FP = ex_1 - a; \qquad F'P = ex_1 + a.$$

To compute the denominators, find where the tangent at P, whose equation is given by (1), cuts the axis of x. Denoting the abscissa of M by x', we have:

$$x' = \frac{a^2}{x_1}.$$

Now, $$FM = OF - OM = c - x',$$

and $$c - x' = c - \frac{a^2}{x_1} = \frac{cx_1 - a^2}{x_1}.$$

But $c = ae$, and so

$$cx_1 - a^2 = a(ex_1 - a).$$

Thus $$c - x' = \frac{a}{x_1}(ex_1 - a),$$

* Let the student prove this proposition as an exercise.

and we arrive finally at the desired expression for FM:

$$FM = \frac{a}{x_1}(ex_1 - a).$$

In a similar manner it is shown that

$$F'M = \frac{a}{x_1}(ex_1 + a).$$

From these evaluations it appears that

$$\frac{FP}{FM} = \frac{a}{x_1} \quad \text{and} \quad \frac{F'P}{F'M} = \frac{a}{x_1}.$$

Hence (2) is a true equation, and the proof is complete for the case that P lies on the right-hand branch. Since, however, the curve is symmetric in the conjugate axis, the theorem is true for the left-hand branch also.

Latus Rectum. The latus rectum of a hyperbola is defined as a chord passing through a focus and perpendicular to the transverse axis. The term is also applied to the length of such a chord.

EXERCISES

1. Find the slope of the hyperbola $4x^2 - y^2 = 15$ at the point $(2, -1)$. *Ans.* -8.

2. Find the equation of the tangent of the hyperbola of Ex. 1 at the point there mentioned. *Ans.* $8x + y = 15$.

3. Find the angle at which the line through the origin bisecting the angle between the positive axes of coördinates cuts the hyperbola of Ex. 1. *Ans.* $30° 58'$.

4. Find the length of the latus rectum of the hyperbola

$$\frac{x^2}{16} - \frac{y^2}{9} = 1. \qquad \textit{Ans. } 4\tfrac{1}{2}.$$

5. Find the length of the latus rectum of the hyperbola of Ex. 1. *Ans.* 15.49.

6. Find the equation of the normal of the hyperbola

$$\frac{x^2}{25} - \frac{y^2}{144} = 1$$

at the extremity of the latus rectum which lies in the first quadrant. *Ans.* $25x + 65y = 2197$.

7. Show that the length of the latus rectum of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is $\frac{2b^2}{a}$.

8. Prove that the tangents at the extremities of the latera recta have slopes $\pm e$.

9. In an ellipse, the focal radii make equal angles with the tangent. Prove this theorem by the method employed in this paragraph to prove the corresponding theorem relating to the hyperbola.

6. New Definition. The Directrices. The locus defined in Ch. VII, § 7, can now be shown to be a hyperbola when $\epsilon > 1$. The analytic treatment given there and in § 8 down to equation (2) and the transformation (3) holds unaltered for the present case.

When, however, $\epsilon > 1$, the new origin, O', lies to the left of O, in the point $\left(-\frac{m}{\epsilon^2 - 1}, 0\right)$, and it is more natural to write (3) in the form

(1) $$x' = x + \frac{m}{\epsilon^2 - 1}, \qquad y' = y,$$

and likewise (4) as

(2) $$x'^2 - \frac{y'^2}{\epsilon^2 - 1} = \frac{\epsilon^2 m^2}{(\epsilon^2 - 1)^2}.$$

This equation passes over into the form

(3) $$\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1,$$

on setting

(4) $$a = \frac{\epsilon m}{\epsilon^2 - 1}, \qquad b = \frac{\epsilon m}{\sqrt{\epsilon^2 - 1}}.$$

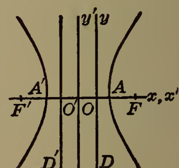

FIG. 12

Thus the locus is seen to be a hyperbola with its center, O', at the point $\left(-\frac{m}{\epsilon^2-1}, 0\right)$, the semi-axes being given by (4).

The value of c is given by the equation $c^2 = a^2 + b^2$. Hence

$$c = \frac{\epsilon^2 m}{\epsilon^2 - 1}. \tag{5}$$

The eccentricity, $e = c/a$, is seen to be precisely ϵ:

$$e = \epsilon,$$

and thus the given constant, ϵ, turns out to be the eccentricity of the hyperbola.

Finally, *F is one of the foci.* For, the distance from O' to F is

$$O'O + OF = \frac{m}{\epsilon^2 - 1} + m = \frac{\epsilon^2 m}{\epsilon^2 - 1},$$

and this, by (5), is precisely c.

The line D is called a *directrix* of the hyperbola. Its distance from the center is

$$O'O = \frac{m}{\epsilon^2 - 1} = \frac{\epsilon m}{\epsilon^2 - 1} \cdot \frac{1}{\epsilon} = \frac{a}{\epsilon}.$$

The Directrices. There is a second directrix, namely, the line D' symmetric to D in the conjugate axis. It is clear from the symmetry of the figure that what is true of the hyperbola with respect to the focus F and the corresponding directrix D is equally true with respect to the focus F' and the directrix D'. Accordingly, the hyperbola is the locus of a point whose distance from a focus bears to its distance from the corresponding directrix a fixed ratio, the eccentricity.

The equations of the directrices of the hyperbola,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

are $\qquad x = \frac{a}{e} \qquad$ and $\qquad x = -\frac{a}{e}.$

EXERCISES

1. Take $\epsilon = 2$ and $m = 3$, the unit of length being 1 cm. With ruler and compasses construct a generous number of points of the locus, and then draw in the locus with a clean, firm line.*

2. Work out the equation of the locus of Ex. 1 directly, using the method of Ch. VII, § 7, but not looking at the formulas. *Ans.* $3x^2 - y^2 + 6x = 9$.

3. By means of a transformation to parallel axes show that the curve of Ex. 2 is a hyperbola whose center is at the point $(-1, 0)$ and whose axes are of lengths 4 and $4\sqrt{3}$.

4. Show that in the general case the distances of the vertices, A and A', from O are:

$$OA = \frac{m}{\epsilon + 1}, \qquad A'O = \frac{m}{\epsilon - 1}.$$

5. Collect the results of this paragraph in a syllabus, arranged in tabular form, giving each of the quantities, a, b, c, $O'O$, OA, $A'O$, OF, and $F'O$, in terms of m and ϵ.

6. Work out each of the quantities of Ex. 5 directly for the curve of Ex. 2 and verify the result by substituting the values $\epsilon = 2$, $m = 3$ in the formulas of the syllabus.

7. Show that the tangent to the hyperbola

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

at an extremity of a latus rectum cuts the transverse axis in the same point in which this axis is cut by a directrix.

8. The same for any hyperbola.

* The footnote of p. 114 applies in the present case with the obvious modification that the distance of the parallel from D must now be *half* the radius of the circle. Moreover, *two* parallels to D must now be drawn, the second one, as soon as the radius has increased sufficiently, giving points on the left-hand branch.

9. Prove directly that, if P is any point of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

the ratio of its distance from a focus to its distance from the corresponding directrix equals the eccentricity.

10. Prove that the ratio of the distance between the foci of a hyperbola to the distance between the directrices equals the square of the eccentricity.

7. The Parabola as the Limit of Hyperbolas. Summary. Equation (3) of Ch. VII, § 7, namely,

$$(1) \qquad (1-\epsilon^2)x^2+y^2-2\,mx+m^2=0,$$

represents a hyperbola when $\epsilon>1$ and a parabola when $\epsilon=1$. If, then, we let ϵ approach 1 through values greater than 1, the hyperbola which (1) represents will approach a parabola as its limiting position.

Suppose, for example, that we take $m=2$ and let ϵ take on successively the values 2, $1\frac{1}{2}$, $1\frac{1}{4}$, $1\frac{1}{8}$, $\cdots$. Drawing the corresponding hyperbolas, we find that, whereas the directrix D and the right-hand focus F are always fixed, the center and the left-hand vertex keep receding to the left, and that their distances from O, namely,

$$O'O=\frac{m}{\epsilon^2-1},\qquad A'O=\frac{m}{\epsilon-1},$$

increase without limit. Thus, when ϵ approaches 1, the left-hand branch of the hyperbola recedes indefinitely to the left and disappears in the limit, whereas, meanwhile, the right-hand branch gradually changes shape and in the limit becomes the parabola whose directrix is D and whose focus is F.

Summary. Let us now combine the results of § 6 with those of § 8, Ch. VII. We have proved that equation (1) represents an ellipse, a parabola, or a hyperbola, according as $\epsilon<1$, $\epsilon=1$, or $\epsilon>1$. In case of the ellipse and the hyperbola the constant ϵ turned out to be the eccentricity e. We are led then to give to the parabola an eccentricity, namely, $\epsilon = e = 1$.

THEOREM. *The locus of a point which moves so that its distance from a fixed point bears to its distance from a fixed line, not passing through the fixed point, a given ratio ϵ is an ellipse, a parabola, or a hyperbola, according as ϵ is less than, equal to, or greater than unity. In every case the constant ϵ equals the eccentricity.*

Since always $\epsilon = e$, we may suppress ϵ in future work, and use e exclusively. Thus equation (1) becomes

(2) $$(1 - e^2)x^2 + y^2 - 2\,mx + m^2 = 0.$$

The theorem furnishes a blanket definition for the ellipse, parabola, and hyperbola, which might have been used instead of the separate definitions which we have given. It should be noted, however, that this blanket definition does not include the circle. For, if we set $e = 0$ in (2), the equation reduces to

$$(x - m)^2 + y^2 = 0,$$

which represents merely the focus $F: (m, 0)$.

The fact that the blanket definition does not yield a circle as a special case in no way discredits the circle as the limiting form of an ellipse when the eccentricity approaches zero, Ch. VII, § 1. The reason that a circle cannot be defined in the new manner is because it has no directrices. When the eccentricity of an ellipse approaches zero, the major axis remaining constant, the distance a/e of the directrices from the center increases indefinitely, so that in the limit, when the ellipse becomes a circle, the directrices have disappeared.*

* It is, of course, possible to obtain the circle as a *limiting* curve approached by ellipses defined in the new way. If the points F and A of Fig. 12, Ch. VII, are held fast and m is allowed to increase indefinitely, then it can be shown that ϵ approaches zero and that a and b both approach the fixed distance AF. Thus the variable ellipse approaches a circle as its limit.

8. Hyperbolas with Foci on the Axis of *y*. Conjugate Hyperbolas. Let the student show that the equation of the hyperbola whose foci are at the points $(0, \pm C)$ on the axis of y and the difference of whose focal radii is $2B$ is

$$\frac{x^2}{A^2} - \frac{y^2}{B^2} = -1,$$

where

$$C^2 = A^2 + B^2.$$

The transverse axis of this hyperbola is the axis of y; the conjugate axis, the axis of x. The length of the major axis is $2B$; that of the minor axis, $2A$. The eccentricity is C/B and the asymptotes have the equations,

$$\frac{x}{A} - \frac{y}{B} = 0 \qquad \text{and} \qquad \frac{x}{A} + \frac{y}{B} = 0.$$

Conjugate Hyperbolas. The two hyperbolas,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{and} \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1,$$

have the same asymptotes. The transverse axis of each is the conjugate axis of the other, and the major axis of each is the minor axis of the other.

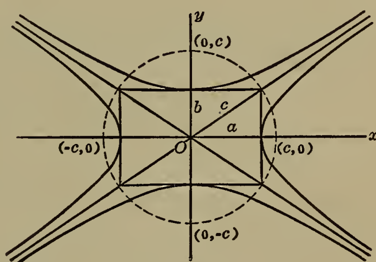

FIG. 13

Taken together, the two hyperbolas form what is called a *pair of conjugate hyperbolas*. The relationship between them is perfect in its duality. We say, then, that each is the *conjugate* of the other.

The two hyperbolas together are tangent externally at their vertices to the rectangle of § 4 at the mid-points of its sides. Moreover, all straight lines through the common center O, except two, meet one hyperbola or the other in two points, and the segment thus terminated is bisected at O.

The student should compare these facts with the corresponding ones concerning a single ellipse and the circumscribed rectangle.

EXERCISES

1. Find the coördinates of the foci, the lengths of the axes, the slopes of the asymptotes, and the value of the eccentricity for each of the hyperbolas:

(*a*) $\frac{x^2}{9}-\frac{y^2}{16}=-1$; (*c*) $y^2-x^2=4$;

(*b*) $5x^2-4y^2+20=0$; (*d*) $3x^2-2y^2+6=0$.

Draw an accurate figure in each case.

2. What are the equations of the hyperbolas conjugate to the hyperbolas of Ex. 1?

3. Find the equation of the hyperbola whose vertices are in the points $(0, \pm 4)$ and whose eccentricity is $\frac{3}{2}$.

Ans. $4x^2-5y^2+80=0$.

4. Find the equation of the hyperbola the extremities of whose minor axis are in the points $(\pm 3, 0)$ and whose eccentricity is $\frac{5}{4}$.

5. Prove that the sum of the squares of the reciprocals of the eccentricities of the two conjugate hyperbolas

$$\frac{x^2}{9}-\frac{y^2}{16}=1, \qquad \frac{x^2}{9}-\frac{y^2}{16}=-1,$$

is equal to unity.

6. Prove the theorem of Ex. 5 for the general pair of conjugate hyperbolas.

7. Show that the foci of a pair of conjugate hyperbolas lie on a circle.

9. Parametric Representation. It is possible to construct a hyperbola, given its axes, AA' and BB', by a method much like that of Ch. VII, § 10, for the ellipse.

Let the two circles, C and C', and the ray from O, be drawn as before. At the point L draw the tangent to C', and mark the point Q where the ray cuts this line. At R draw the tangent to C and mark the point S where this tangent cuts the axis of x.

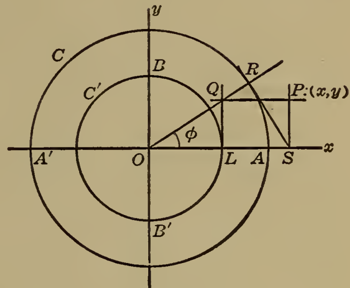

FIG. 14

The locus of the point $P:(x, y)$, in which the parallel to the axis of x through Q and the parallel to the axis of y through S intersect, is the hyperbola.

For, $\qquad OR = a, \qquad OL = b,$

and $\qquad x = OS = a \sec\phi, \qquad y = LQ = b \tan\phi.$

Hence $\qquad \dfrac{x}{a} = \sec\phi, \qquad \dfrac{y}{b} = \tan\phi,$

and since $\qquad \sec^2\phi - \tan^2\phi = 1,$

it follows that $\qquad \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1.$

Conversely, any point (x, y) whose coördinates satisfy this equation is seen to lead to an angle ϕ, for which the above formulas hold.

We thus obtain the following parametric representation of the hyperbola:

$$x = a \sec\phi, \qquad y = b \tan\phi.$$

The circle C, constructed on the major axis of the hyperbola as a diameter, is known as the *auxiliary circle* of the hyperbola, and the angle ϕ is called the *eccentric angle.*

EXERCISES

1. Carry out the construction described above for the cases:

(a) $\qquad a = 3$ cm., $\qquad b = 2$ cm.

(b) $a = 3$ cm., $b = 3$ cm.

(c) $a = 2$ cm., $b = 3$ cm.

2. Obtain a parametric representation of the hyperbola

$$\frac{x^2}{A^2} - \frac{y^2}{B^2} = -1.$$

10. Conic Sections. The ellipse (inclusive of the circle), the hyperbola, and the parabola are often called *conic sections*, because they are the curves in which a cone of revolution is cut by planes.

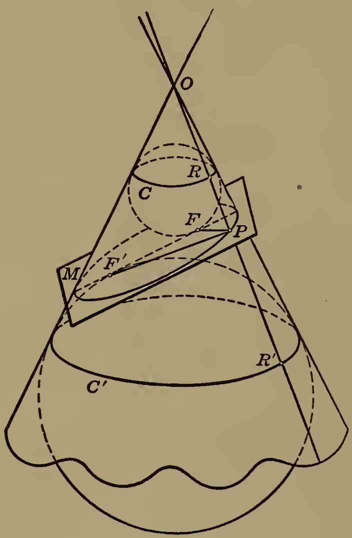

Fig. 15

Suppose a plane M cuts only one nappe of the cone, as is shown in the accompanying drawing. Let a small sphere be placed in the cone near O, tangent to this nappe along a circle. It will not be large enough to reach to the plane M. Now let the sphere grow, always remaining tangent to the cone along a circle. It will finally just reach the plane. Mark the point of tangency, F, of the plane M with the sphere, and also the circle of contact, C, of the sphere with the cone.

As the sphere grows still larger, it cuts the plane M, but finally passes beyond on the other side. In its last position, in which it still meets M, it will be tangent to M. Let the point of tangency be denoted by F', and the circle of contact of the sphere with the cone by C'.

Through an arbitrary point P of the curve of intersection of M with the cone passes a generator OP of the cone; let it cut C in R and C' in R'. Then RR', being the slant height of the frustum * cut from the cone by the planes of C and C', is of the same length, $2a$, for all points P.

Join P with F. Then PF and PR, being tangents from P to the same sphere, are equal. Similarly, PF' and PR' are equal. Hence

$$FP + F'P = RP + R'P = RR',$$

or

$$FP + F'P = 2a.$$

But this locus is by definition an ellipse with its foci at F and F', and hence the proposition is proved for the case that M cuts only one nappe, the intersection being a closed curve.

If the plane M cuts both nappes, but does not pass through O, it is a little harder to draw the figure, one sphere being inscribed in the one nappe, the other, in the other nappe. A similar study shows that here the *difference* between FP and $F'P$ is equal to RR', and hence the locus is a hyperbola.

The parabola corresponds to the case that M meets only one nappe, but does not cut it in a closed curve. This case is realized when M does not pass through O and is parallel to a generator of the cone.

Let L be a line which is perpendicular to the axis of the cone in a point of the axis distinct from the vertex. As a plane, M, rotates about L, it will cut from the cone all three kinds of conics. This will still be true if we take, as L, any line of space which does not pass through the vertex and is not parallel to a generator.

11. Confocal Conics. Two conics are said to be *confocal* if they have the same foci; in the case of two parabolas, we demand, further, that they have the same axis.

* No technical knowledge of Solid Geometry beyond the definitions of the terms used (which can be found in any dictionary) is here needed. On visualizing the figure, the truth of the statements regarding the space relations becomes evident.

Consider an ellipse and a hyperbola which are confocal. They evidently intersect in four points.*

Let P be one of these points. Join P with F and F'. Then FP and $F'P$ are focal radii both of the ellipse and of the hyperbola. Now, the tangent to a hyperbola at any point not a vertex bisects the angle between the focal radii drawn to that point, § 5; and the normal to an ellipse at any point not on the transverse axis bisects the angle between the focal radii drawn to that point, Ch. VII, § 4. It follows, then, that the tangent to the hyperbola at P and the normal to the ellipse at this point coincide. Hence the two curves intersect at right angles, or *orthogonally*, as we say. We have thus proved the following

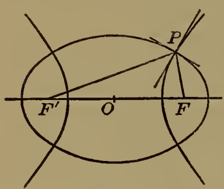

Fig. 16

Theorem. *A pair of confocal conics, one of which is an ellipse and the other a hyperbola, cut each other orthogonally.*

Confocal Parabolas. Consider two parabolas having the same focus and the same axis. If both open out in the same direction, they have no point in common. If, however, they open out in opposite directions, they intersect in two points which are symmetrically situated with respect to the axis.

In the latter case, the parabolas intersect orthogonally, as has already been proved analytically; cf. Ch. VI, § 3, Ex. 10.

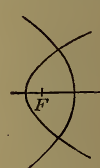

Fig. 17

This result could have been forecast, as a consequence of the relations established in § 7. For, if one focus, F, and the two corresponding directrices of a pair of confocal conics, consisting of an ellipse and a hyperbola, are held fast, and if the other focus is made to recede indefinitely, each of the conics approaches a parabola. But the

* Let the student satisfy himself that two confocal ellipses do not intersect, and that the same is true of two confocal hyperbolas.

conics always intersect orthogonally, and so the same will be true of the limiting curves, the parabolas.

To obtain a prescribed pair of parabolas, like those described above, as limiting curves, it is necessary merely to choose the two confocal conics so that the directrices corresponding to F are at the proper distances from F.

Mechanical Constructions. It is possible to draw with ease a large number of confocal ellipses by the method set forth in Ch. VII, § 1. Let thumb tacks be inserted at F and F', but not pushed clear down. Let a thread be tied to the tack at F, passed round the tack at F', and held fast at M. Then an ellipse can be drawn with F and F' as foci.

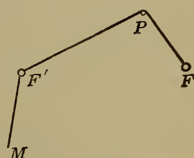

Fig. 18

Now let the thread be unwound at F' and drawn in or paid out slightly, so that the length of the free thread between F and F' is changed. On repeating the above construction, a second ellipse with its foci at F and F' is obtained; and so on.

There is an analogous construction for a hyperbola, which has not yet been mentioned. Tie a thread to a pencil point,* pass the thread round the pegs at F and F' as shown, hold the free ends firmly together at M, and, keeping the thread taut by pressing on the pencil, allow M to move. The pencil then obviously traces out a hyperbola.

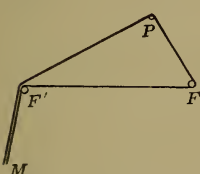

Fig. 19

By pulling one end of the thread in slightly at M, or by paying it out, and then repeating the construction, a new hyperbola with the same foci is obtained; and so on.

Parabolas. The accompanying figure suggests a means for drawing a parabola mechanically.

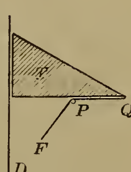

Fig. 20

* To keep the thread from slipping off, cut a groove in the lead, such as would be obtained if the pencil were turned about its axis in a lathe and the point of a chisel were held against the lead close to the wood.

A ruler, D, is held fast and a triangle, T, is allowed to slide along the ruler. A thread is tied at F and Q, and a pencil point, P, keeps the thread taut and pressed against the triangle.

EXERCISES

1. Show that the conics,

$$\frac{x^2}{24}+\frac{y^2}{8}=1 \quad \text{and} \quad \frac{x^2}{4}-\frac{y^2}{12}=1,$$

are confocal.

2. Prove that the equation,

$$\frac{x^2}{9+\lambda}+\frac{y^2}{5+\lambda}=1,$$

represents an ellipse for each value of λ greater than -5 and represents a hyperbola for each value of λ between -9 and -5. Show that all these ellipses and hyperbolas are confocal, with the points $(\pm 2, 0)$ as foci.

3. For what values of λ does the equation

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,$$

where a and b are given positive constants such that $a > b$, represent i) ellipses? ii) hyperbolas? Show that all these conics are confocal.

4. Draw a set of confocal ellipses and hyperbolas.

5. Draw a set of confocal parabolas, all having the same transverse axis, some opening in one direction, some in the other.

EXERCISES ON CHAPTER VIII

1. The axes of a hyperbola which goes through the points $(1, 4)$, $(-2, 7)$ are the axes of coördinates. Find the equation of the hyperbola. *Ans.* $y^2-11x^2=5$

2. Show that the hyperbolas defined by the equation

$$4x^2-5y^2=c,$$

where c is an arbitrary constant, not zero, all have the same asymptotes.

3. How many hyperbolas are there with the lines

$$3x^2 - 16y^2 = 0$$

as asymptotes? Find an equation which represents them all.

Ans. $3x^2 - 16y^2 = c, \quad c \neq 0.$

4. What is the equation of all the rectangular hyperbolas with the axes of coördinates as axes?

5. A hyperbola with the lines $4x^2 - y^2 = 0$ as asymptotes goes through the point (1, 1). What is its equation?

Ans. $4x^2 - y^2 = 3.$

6. The asymptotes of a hyperbola go through the origin and have slopes ± 2. The hyperbola goes through the point (1, 3). Find its equation. *Ans.* $4x^2 - y^2 = -5.$

7. The two hyperbolas of Exs. 5 and 6 have the same asymptotes, but lie in the opposite pairs of regions into which the plane is divided by the asymptotes. Show that the sum of the squares of the reciprocals of their eccentricities equals unity.

8. Prove that of the hyperbolas of Ex. 2 those for which c is positive are all similar, and that this is true also of those for which c is negative. If e is the common value of the eccentricity of the hyperbolas of the first set and e' is that of the hyperbolas of the second set, show that

$$\frac{1}{e^2} + \frac{1}{e'^2} = 1. \tag{1}$$

9. Prove that the relation (1) is valid for the eccentricities of any two hyperbolas which have the same asymptotes but lie in the opposite regions between the asymptotes.

10. Show that two hyperbolas which are related as those described in the previous exercise have the same eccentricity if and only if they are rectangular hyperbolas.

11. A hyperbola with its center in the origin has the eccentricity 2. Find the equations of the asymptotes, (*a*) if the foci lie on the axis of x; (*b*) if the foci lie on the axis of y.

Ans. (*a*) $3x^2 - y^2 = 0$; (*b*) $x^2 - 3y^2 = 0$.

12. What is the equation representing all the hyperbolas which have their centers in the origin and eccentricity 2, (*a*) if the foci lie on the axis of x? (*b*) if the foci lie on the axis of y? Show that in either case the vertices lie midway between the center and the foci.

13. Prove that the vertices of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

subtend a right angle at each of the points $(0, \pm b)$ when and only when the hyperbola is rectangular. What is the corresponding theorem in the case of the ellipse?

14. The projections of a point P of a hyperbola on the transverse and conjugate axes are P_1 and P_2. The tangent at P meets these axes in T_1 and T_2. Show that $OP_1 \cdot OT_1 = a^2$ and $OP_2 \cdot OT_2 = -b^2$, where O is the center of the hyperbola and a and b are the semi-axes.

15. Prove that the segment of a tangent to a hyperbola between the point of contact and a directrix subtends a right angle at the corresponding focus.

16. The projection of a point P of a hyperbola on the transverse axis is P_1 and the normal at P meets this axis at N_1. Show that the ratio of the distances of the center from N_1 and P_1 equals the square of the eccentricity.

17. Prove that the line joining a point P of a hyperbola with the center and the line through a focus perpendicular to the tangent at P meet on a directrix.

18. Find the equation of the circle which is tangent to a hyperbola at the upper ends of the two latera recta.

19. Let O be the center, A a vertex, and F the adjacent focus of a hyperbola. The tangent at a point P meets the transverse axis at T and the tangent at A meets OP at V. Show that TV is parallel to AP.

20. Show that an asymptote, a directrix, and the line through the corresponding focus perpendicular to the asymptote go through a point.

21. A line through a focus F parallel to an asymptote meets the hyperbola at P. Show that the tangent at P, the other asymptote, and the line of the latus rectum through F meet in a point.

22. Let F be a focus and D the corresponding directrix of a hyperbola. A line through a point P of the hyperbola parallel to an asymptote meets D in the point K. Prove that the triangle FPK is isosceles.

Exercises 23–33. In proving the theorems in these exercises it will, in general, be necessary to make actual use of the equation expressing the fact that a certain point lies on the hyperbola.

23. The tangent to a hyperbola at a point P meets the tangent at one vertex in Q. Prove that the line joining the other vertex to P is parallel to the line joining the center to Q.

24. Let F be a focus and D the corresponding directrix of a hyperbola. Prove that the segment cut from D by the lines joining the vertices with an arbitrary point on the hyperbola subtends a right angle at F.

25. Prove that the product of the distances of the foci of a hyperbola from a tangent is constant, *i.e.* independent of the choice of the tangent.

26. Let A and A' be the vertices of a rectangular hyperbola and let P and P' be two points of the hyperbola symmetric in the transverse axis. Prove that AP is perpendicular to $A'P'$ and that AP' is perpendicular to $A'P$.

27. Show that the product of the focal radii to a point on a rectangular hyperbola is equal to the square of the distance of the point from the center.

28. Prove that the angles subtended at the vertices of a rectangular hyperbola by a chord parallel to the conjugate axis are supplementary.

29. Prove that the product of the distances of an arbitrary point on a hyperbola from the asymptotes is constant, *i.e.* the same for every choice of the point.

30. A line through an arbitrary point P on a hyperbola parallel to the conjugate axis meets the asymptotes in M and N. Show that the product of the segments in which P divides MN is constant.

31. Prove that the segment of a tangent to a hyperbola cut out by the asymptotes is bisected by the point of contact of the tangent.

32. Show that the tangent to a hyperbola at an arbitrary point forms with the asymptotes a triangle which has a constant area.

33. The tangent to a hyperbola at a point P meets the tangents at the vertices in M and N. Prove that the circle on MN as a diameter passes through the foci.

Loci

34. Find the locus of a point whose distance from a given circle always equals its distance from a given point without the circle. First give a geometric construction, with ruler and compass, for points on the locus. Then find the equation of the locus.

35. The base of a triangle is fixed and the product of the tangents of the base angles is a negative constant. What is the locus of the vertex?

36. A line moves so that the area of the triangle which it forms with two given perpendicular lines is constant. Find the locus of the mid-point of the segment cut from it by these lines.

Ans. Two conjugate rectangular hyperbolas, with the given lines as asymptotes.

37. Given a fixed line L and a fixed point A, not on L. A point P moves so that its distance from L always equals the distance AQ, where Q is the foot of the perpendicular dropped from P on L. What is the locus of P?

38. What is the locus of the point P of the preceding exercise, if the ratio of its distance from L to the distance AQ is constant?

CHAPTER IX

CERTAIN GENERAL METHODS

1. Tangents. Let it be required to find the tangent line to a given curve at an arbitrary point.

In the case of the circle the tangent is perpendicular to the radius drawn to the point of tangency. But this solution is of so special a nature that it suggests no general method of attack. A general method must be based on a general property of tangents, irrespective of the special curve considered. Such a method is the following. Let P be an arbitrary point of a given curve, C, at which it is desired to draw the tangent, T. Let a second point, P', be chosen on C, and draw the secant, PP'. As P' moves along C and approaches the fixed point P as its limit, the secant rotates about P as a pivot and approaches the tangent, T, as its limiting position. *Thus the tangent appears as the limit of the secant.*

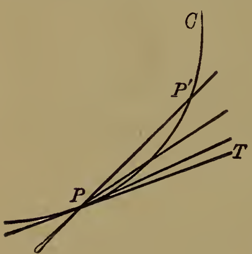

Fig. 1

If, now, in a given case we can find an expression for the *slope* of the secant, the limit approached by this expression will give us the *slope of the tangent.* The slope of the tangent to the curve at P we shall call, for the sake of brevity, the *slope of the curve* at P.

Example 1. Find the slope of the curve

$$y = x^2 \tag{1}$$

at a given point, P.

Let the coördinates of P be (x_1, y_1); those of P', (x', y'), or $(x_1 + h, y_1 + k)$. Then

$$PQ = h, \qquad QP' = k,$$

and we have, for the slope of the secant PP', the expression:

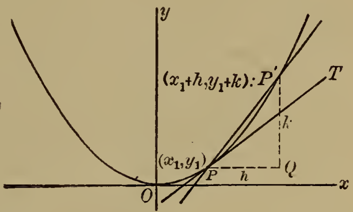

FIG. 2

(2) $$\tan \tau' = \frac{k}{h},$$

where $\tau' = \measuredangle QPP'$. The slope of the tangent line, T, at P is, then,

(3) $$\tan \tau = \lim_{P' \doteq P} \tan \tau' = \lim_{h \doteq 0} \frac{k}{h},$$

where $\tau = \measuredangle QPT$. The sign $\doteq$ is used to mean "approaches as its limit," and the expression: $\lim\limits_{P' \doteq P} \tan \tau'$, is read: "the limit of $\tan \tau'$, as P' approaches P."

Suppose, for example, that P is the point (1, 1). Let us compute k and $\tan \tau'$ for a few values of h. Here, $x_1 = 1$ and $y_1 = 1$. If $h = .1$, then

$$x' = x_1 + h = 1.1,$$
$$y' = y_1 + k = (1.1)^2 = 1.21,$$
$$k = .21,$$

and hence $$\tan \tau' = \frac{.21}{.1} = 2.1.$$

Next, let P' be the point for which

$$x' = 1.01.$$

Then $$y' = 1.0201,$$

$$h = .01, \qquad k = .0201,$$

and hence $$\tan \tau' = \frac{.0201}{.01} = 2.01.$$

Let the student work out one more case, taking $x' = 1.001$. He will find that here $k = .002001$ and

$$\tan \tau' = 2.001.$$

These results can be presented conveniently in the form of a table:

h	k	$\tan \tau' = \frac{k}{h}$
.1	.21	2.1
.01	.0201	2.01
.001	.002001	2.001

The numbers in the last column appear to be approaching nearer and nearer to the limit 2; in other words, the slope of the curve in the point (1, 1) appears to be 2. Let us prove that this is actually the case. Since the proof is just as simple for an arbitrary point P, we will return to the general case.

The point P being a point of the curve (1), its coördinates (x_1, y_1) must satisfy that equation. Hence

(4) $$y_1 = x_1^2.$$

Similarly, for the point P' whose coördinates are $(x_1 + h, y_1 + k)$:

$$y_1 + k = (x_1 + h)^2,$$

or

(5) $$y_1 + k = x_1^2 + 2\,x_1 h + h^2.$$

Subtracting (4) from (5), we get:

$$k = 2\,x_1 h + h^2.$$

Consequently, $$\tan \tau' = \frac{k}{h} = 2\,x_1 + h.$$

Now let P' approach P; h will then approach 0, and we shall have

$$\lim_{P' \doteq P} \tan \tau' = \lim_{h \doteq 0} \frac{k}{h} = \lim_{h \doteq 0} (2\,x_1 + h).$$

But

$$\lim_{P' \doteq P} \tan \tau' = \tan \tau, \qquad \text{and} \qquad \lim_{h \doteq 0} (2\,x_1 + h) = 2\,x_1.$$

Hence $$\tan \tau = 2\,x_1.$$

We can say, then, that *the slope of the curve* (1), *at an arbitrary point* $P:(x_1, y_1)$ *on it, is*

$$\lambda = 2x_1.$$

If, in particular, P is the point (1, 1), the slope of the tangent there is $\lambda = 2 \cdot 1 = 2$, and thus the indication given by the above table is seen to be borne out.

Example 2. Find the slope of the curve

$$y = \frac{a^2}{x} \tag{6}$$

at an arbitrary point $P:(x_1, y_1)$ of the curve.

Denote, as before, the coördinates of a second point, P', by

$$x' = x_1 + h, \qquad y' = y_1 + k.$$

Then, since P and P' lie on the curve,

$$y_1 = \frac{a^2}{x_1}$$

and

$$y_1 + k = \frac{a^2}{x_1 + h}.$$

Hence

$$k = \frac{a^2}{x_1 + h} - \frac{a^2}{x_1}.$$

Nothing is more natural than to reduce the right-hand side of this equation to a common denominator. Thus

$$k = \frac{-a^2h}{x_1(x_1 + h)}.$$

Consequently,

$$\tan\tau' = \frac{k}{h} = \frac{-a^2}{x_1(x_1 + h)}.$$

We are now ready to let P' approach P:

$$\lim_{P' \doteq P} \tan\tau' = \lim_{h \doteq 0} \frac{-a^2}{x_1(x_1 + h)}.$$

The limit approached by the right-hand side is obviously $-a^2/x_1^2$, and so

$$\tan\tau = -\frac{a^2}{x_1^2}.$$

We have, then, as the final result: *The slope of the curve* (6), *at an arbitary point* (x_1, y_1) *on it, is*

$$\lambda = -\frac{a^2}{x_1^2}.$$

Equation of the Tangent. Since the tangent to the curve (1),

$$y = x^2,$$

at the point (1, 1) has the slope 2, its equation is

$$y - 1 = 2(x - 1), \qquad \text{or} \qquad 2x - y - 1 = 0.$$

Similarly, the equation of the tangent to the curve (1) at an arbitrary point $P:(x_1, y_1)$ is

$$y - y_1 = 2x_1(x - x_1),$$

or

$$y - y_1 = 2x_1x - 2x_1^2.$$

This equation may be simplified by use of the equality,

$$y_1 = x_1^2,$$

which says that the point P lies on the curve. For, if we replace the term $2x_1^2$ by its equal, $2y_1$, and then combine the terms in y_1, the equation becomes

$$y + y_1 = 2x_1x.$$

This equation of the tangent is of the first degree in x and y, as it should be. The quantities x_1 and y_1 are the arbitrary, but in any given case *fixed*, coördinates of P and are not variables.

Equation of the Normal. The line through a point P of a curve perpendicular to the tangent at P is known as the *normal* to the curve at P.

Since the tangent to the curve $y = x^2$ at the point (1, 1) has the slope 2, the normal at this point has the slope $-\frac{1}{2}$. Consequently, the equation of the normal is

$$y - 1 = -\tfrac{1}{2}(x - 1), \qquad \text{or} \qquad x + 2y - 3 = 0.$$

EXERCISES

1. Determine the slope of the curve $y = x^2 - x$ at the point (3, 6). First make out a table like that under Example 1, and hence infer the probable slope. Then take an arbitrary point (x_1, y_1) on the curve and determine the actual slope at this point by finding

$$\lim_{h \doteq 0} \frac{k}{h}.$$

2. The same for the curve $8y = 3x^3$ at the point (2, 3).

3. The same for the curve $y = 2x^2 - 3x + 1$ at the point (1, 0).

Find the slope of each of the following curves at an arbitrary point $P:(x_1, y_1)$. No preliminary study of a numerical case, like that which gave rise to the table under Example 1, is here required.

4. $y = x^2 - 3x + 1$. *Ans.* $\lambda = 2x_1 - 3$.

5. $y = 2x^2 - x - 4$.

6. $y = x^3 - x$.

7. $y = 4x^3 - 2x^2 + 5$.

8. $y = x^3 + x^2 + x + 1$.

9. $y = x^3 + px + q$. *Ans.* $\lambda = 3x_1^2 + p$.

10. $y = x^4 - a^4$. *Ans.* $\lambda = 4x_1^3$.

11. $y = \dfrac{1}{x^2}$. *Ans.* $\lambda = -\dfrac{2}{x_1^3}$.

12. $y = \dfrac{a^4}{x^3}$.

13. $y = \dfrac{1}{1-x}$. *Ans.* $\lambda = \dfrac{1}{(1-x_1)^2}$.

14. $y = \dfrac{2x}{3x-4}$. *Ans.* $\lambda = -\dfrac{8}{(3x_1-4)^2}$.

15. $y = ax^2 + bx + c$. *Ans.* $\lambda = 2ax_1 + b$.

16. $y = ax^3 + bx^2 + cx + d$.

17. $y = x^n$, (n, a positive integer) *Ans.* $\lambda = nx_1^{n-1}$.

18. $y = cx^n$.

Find the equations of the tangents to the following curves at the points specified. In each case reduce the equation obtained to the simplest form.

19. The curve of Ex. 1 at the points (3, 6); (x_1, y_1).

Ans. $5x - y - 9 = 0$; $(2x_1 - 1)x - y - x_1^2 = 0$

20. The curve of Ex. 3 at the points (x_1, y_1); (1, 0).

21. The curve of Ex. 4 at the points (x_1, y_1); (−1, 5)

22. The curve of Ex. 11 at the points (1, 1); (x_1, y_1).

23. The curve of Ex. 17 at the point (x_1, y_1).

Ans. $nx_1^{n-1}x - y - (n-1)y_1 = 0$.

24. The curve of Ex. 13 at the point whose abscissa is 2.

25. The curve of Ex. 14 at the point whose abscissa is 4.

26. Find the equations of the normals to the curves of Exs. 21, 22 at the designated points.

2. Continuation. Implicit Equations. We have applied the general method to curves whose equations are given in the form: $y =$ a simple expression in x. More precisely, this "simple expression" has each time been a polynomial (or even a monomial), or the ratio of two such expressions.

But even the simplest forms of the equations of the conics are, as a rule, such that, if the equation be solved for y, radicals will appear. In such cases, the following method of treatment can be used with advantage.

The Parabola. Let it be required to find the slope of the parabola

$$y^2 = 2mx \tag{1}$$

at any point $P: (x_1, y_1)$ on the curve.

We will treat first a numerical case, setting $m = 2$:

$$y^2 = 4x. \tag{2}$$

Since P is on the curve, we have

$$y_1^2 = 4x_1. \tag{3}$$

Since $P' : (x_1 + h,\ y_1 + k)$ is also on the curve, we have:

$$(y_1 + k)^2 = 4(x_1 + h),$$

or

(4) $$y_1^2 + 2\,y_1 k + k^2 = 4\,x_1 + 4\,h.$$

Subtract (3) from (4):

$$2\,y_1 k + k^2 = 4\,h.$$

Divide this equation through by h, to obtain an equation for $\tan\tau' = k/h$:

$$2\,y_1\frac{k}{h} + k\frac{k}{h} = 4, \qquad \text{or} \qquad 2\,y_1 \tan\tau' + k\tan\tau' = 4.$$

Solve the latter equation for $\tan\tau'$:

$$\tan\tau' = \frac{4}{2\,y_1 + k}.$$

We are now ready to let P' approach P as its limit. This means that h and k both approach 0. We have, then,

$$\lim_{P' \doteq P} \tan\tau' = \lim_{h \doteq 0} \frac{4}{2\,y_1 + k},$$

or

$$\tan\tau = \frac{4}{2\,y_1} = \frac{2}{y_1}.$$

It has been tacitly assumed that $y_1 \neq 0$. If $y_1 = 0$, then $\tan\tau'$ increases indefinitely as h, and with it k, approaches zero. Thus the tangent line is seen to be perpendicular to the axis of x at this point, as obviously is, in fact, the case, since the point is the vertex of the parabola.

The student will now carry through by himself the corresponding solution in the general case of equation (1). He will arrive at the result: *The slope λ of the parabola*

$$y^2 = 2\,mx$$

at an arbitrary point (x_1, y_1) of the curve is

(5) $$\lambda = \frac{m}{y_1}.$$

The Ellipse. The treatment in the case of the ellipse,

$$(6) \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

is precisely similar. Writing (6), for convenience, in the form

$$(7) \qquad b^2x^2+a^2y^2=a^2b^2,$$

we are led to the following equations *:

$$(8) \qquad b^2x_1^2+a^2y_1^2=a^2b^2;$$

$$b^2(x_1+h)^2+a^2(y_1+k)^2=a^2b^2,$$

or

$$(9) \qquad b^2x_1^2+a^2y_1^2+2\,b^2x_1h+2\,a^2y_1k+b^2h^2+a^2k^2=a^2b^2.$$

Subtract (8) from (9):

$$2\,b^2x_1h+2\,a^2y_1k+b^2h^2+a^2k^2=0.$$

Divide by h:

$$2\,b^2x_1+2\,a^2y_1\frac{k}{h}+b^2h+a^2k\frac{k}{h}=0,$$

or

$$2\,b^2x_1+2\,a^2y_1\tan\tau'+b^2h+a^2k\tan\tau'=0.$$

Solve this equation for $\tan\tau'$:

$$\tan\tau'=-\frac{2\,b^2x_1+b^2h}{2\,a^2y_1+a^2k}.$$

Now let P' approach P as its limit:

$$\lim_{P'\doteq P}\tan\tau'=\lim_{h\doteq 0}-\frac{2\,b^2x_1+b^2h}{2a^2y_1+a^2k}.$$

Hence

$$\tan\tau=-\frac{2\,b^2x_1}{2\,a^2y_1}=-\frac{b^2x_1}{a^2y_1}.$$

We have thus obtained the result: *The slope λ of the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

at an arbitrary one of its points (x_1, y_1) is

$$(10) \qquad \lambda=-\frac{b^2x_1}{a^2y_1}.$$

* The student will do well to paraphrase the text at this point with a numerical case, — say, $4\,x^2+9\,y^2=36$.

The Hyperbola. The treatment is left to the student. The result is as follows.

The slope λ *of the hyperbola*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at an arbitrary one of its points (x_1, y_1) *is*

$$\lambda = \frac{b^2 x_1}{a^2 y_1}. \tag{11}$$

Equation of the Tangent. Since the slope of the ellipse at the point (x_1, y_1) is $-b^2x_1/a^2y_1$, the equation of the tangent at (x_1, y_1) is

$$y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1)$$

or, after clearing of fractions and rearranging terms,

$$b^2x_1x + a^2y_1y = b^2x_1^2 + a^2y_1^2.$$

If we divide both sides of this equation by a^2b^2, we have

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}.$$

But, since the point (x_1, y_1) lies on the ellipse, it follows that

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1,$$

and the equation of the tangent becomes

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1.$$

The equation of the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_1, y_1) *is*

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1. \tag{12}$$

In a similar manner let the student establish the equations of the tangents to the hyperbola and the parabola.

The tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point (x_1, y_1) *has the equation*

(13) $$\frac{x_1 x}{a^2} - \frac{y_1 y}{b^2} = 1.$$

The tangent to the parabola

$$y^2 = 2mx$$

at the point (x_1, y_1) *has the equation*

(14) $$y_1 y = m(x + x_1).$$

EXERCISES

Find the slope of each of the following six curves at an arbitrary one of its points, applying each time the method set forth in the text.

1. $2x^2 + 3y^2 = 12$.
2. $x^2 - 4y^2 = 4$.
3. $y^2 = 12x$.
4. $x^2 - y^2 = a^2$.
5. $Ax^2 + By^2 = C$, where A, B, C are all positive.
6. $y^2 = Ax + B$, where $A \neq O$.
7. Find the slope of the parabola $y^2 + 2y = 6x$ at the point (x_1, y_1). *Ans.* $\lambda = \dfrac{3}{y_1 + 1}$.
8. What is the slope of the parabola of Ex. 7 at the origin? *Ans.* 3.
9. Find the slope of the curve

$$x^3 - y^2 - 3x + 4y = 0$$

at the origin. *Ans.* $\lambda = \frac{3}{4}$.

Suggestion. First find the slope at an arbitrary point (x_1, y_1). Then substitute in the result the coördinates of the origin.

10. What angle does the curve

$$2x^3 - 3y^2 + x - y + 1 = 0$$

make with a parallel to the axis of x at the point $(1, 1)$? *Ans.* 45°.

11. Find the slope of the curve $xy = a^2$ at any point (x_1, y_1) by the method of the present paragraph, and show that your result agrees with that of § 1, Example 2.

Find the equation of the tangent to each of the following curves at the point designated, applying each time the method of the text. Reduce the equation to its simplest form.

12. The curve of Ex. 1 at the point (x_1, y_1).
Ans. $2x_1x + 3y_1y = 12$.

13. The curve of Ex. 3 at the points (x_1, y_1); $(3, -6)$.

14. The curve of Ex. 5 at the point (x_1, y_1).
Ans. $Ax_1x + By_1y = C$.

15. The curve of Ex. 6 at the point (x_1, y_1).

16. The curve of Ex. 7 at the points (x_1, y_1); $(\frac{1}{2}, 1)$.

17. The curve of Ex. 9 at the origin.

18. Find the equations of the normals to the curves of Exs. 12, 13 at the points specified.

3. The Equation $u + kv = 0$. Consider the following example.

The equations

(1) $$x + y - 2 = 0,$$

(2) $$x - y = 0,$$

represent two straight lines intersecting in the point (1, 1), as shown in Fig. 3. What can we say concerning the curve*

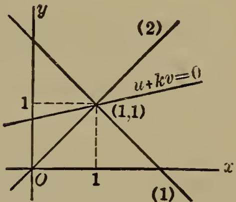

FIG. 3

(3) $$(x + y - 2) + k(x - y) = 0,$$

where k denotes a constant number?

This curve is a straight line, since (3) is an equation of the first degree in x and y. Suppose, now, that various different values are given to k. Then (3) represents various straight lines in turn. What do all these lines have in common?

* The word "curve" is used here in the sense common in analytic geometry, to denote merely the "locus of the equation." Consequently a *curve* in this sense is not necessarily crooked; it may be a straight line.

Since the lines (1) and (2) intersect in the point (1, 1), the coördinates of this point make the left-hand sides of equations (1) and (2), namely, the expressions,

$$x + y - 2 \qquad \text{and} \qquad x - y,$$

vanish. Consequently, they always make the left-hand side of equation (3) vanish. In other words, *equation* (3) *is satisfied by the coördinates of the point of intersection of the lines* (1) *and* (2), NO MATTER WHAT VALUE k HAS. This means that *all* the straight lines represented by (3) go through the point of intersection of the lines (1) and (2).

The result can be restated in the following form. Let the single letter u stand for the whole expression $x + y - 2$:

$$u \equiv x + y - 2,$$

the sign $\equiv$ meaning *identically equal, i.e.* equal, no matter what values x and y have. Similarly, let v stand for $x - y$:

$$v \equiv x - y.$$

Then (3) takes on the form:

(4) $$u + kv = 0.$$

We now restate our result.

If $u = 0$ *and* $v = 0$ *are the equations of two intersecting straight lines, then the equation*

$$u + kv = 0$$

represents a straight line which goes through the point of intersection of the two given lines.

By giving to k a suitable value, $u + kv = 0$ can be made to represent any desired line through the point of intersection (x_1, y_1) of the given lines, with the sole exception of the line $v = 0$. For, let L be the desired line, and let (x_2, y_2) be a point of L distinct from (x_1, y_1). Then, on substituting for x and y the values x_2 and y_2 in the equation $u + kv = 0$, we obtain an equation, in which k is the unknown. This equation can be solved for k, since v does not vanish for the point (x_2, y_2).

Example. Find the equation of the line L which goes through the point of intersection of the lines (1) and (2) and cuts the axis of y in the point $(0, -4)$.

The required line, L, is one of the lines (3); *i.e.* for a suitable value of k, (3) will represent L. To find this value of k, we demand that (3) contain the given point $(0, -4)$ of L. We have, then, setting $x = 0$ and $y = -4$ in (3):

$$(0-4-2)+k(0+4)=0 \qquad \text{or} \qquad k=\tfrac{3}{2}.$$

Consequently, the equation of the line L is

$$x+y-2+\tfrac{3}{2}(x-y)=0 \qquad \text{or} \qquad 5x-y-4=0.$$

That the line represented by the latter equation does actually go through the points $(1, 1)$ and $(0, -4)$ can be verified directly.

The principle which has been set forth for two straight lines evidently applies to any two intersecting curves whatever, so that we are now in a position to state the following general theorem.

THEOREM 1. *Let $u = 0$ and $v = 0$ be the equations of any two intersecting curves. Then the equation*

$$u + kv = 0, \qquad\qquad k \neq 0,$$

represents, in general, a curve which passes through all the points of intersection of the two given curves, and has no other point in common with either of them.*

The last statement in the theorem is new. To prove it, we have but to note that, if the coördinates of a point P satisfy the equation $u + kv = 0$ and also, for example, $v = 0$, they must satisfy the equation $u = 0$; that is, if P is a point on the curve $u + kv = 0$, which lies on one of the given curves, it lies also on the other and so is a point of intersection of the two.

* It may happen in special cases that the locus $u + kv = 0$ reduces to a point, as when, for example,

$$u = 2x^2 + 2y^2 - x, \qquad v = x^2 + y^2 - x, \qquad k = -1.$$

Suppose, now, that the equations $u = 0$ and $v = 0$ represent two curves which have no point of intersection. It follows, then, from the argument just given, that the curve

$$u + kv = 0 \qquad k \neq 0$$

has no point in common with either of the given curves. But it may happen, in this case, that there are no points at all whose coördinates satisfy the equation $u + kv = 0$. Thus, if

$$u = x^2 + y^2 - 1,$$
$$v = x^2 + y^2 - 4,$$

and $k = -1$, we have

$$u + kv \equiv 3,$$

and there are no points whose coördinates satisfy the equation $3 = 0$.

The general result can be stated as

THEOREM 2. *Let $u = 0$ and $v = 0$ be the equations of two non-intersecting curves. Then the equation*

$$u + kv = 0, \qquad k \neq 0,$$

*represents, in general, a curve not meeting either of the two given curves. In particular, it may happen that the equation has no locus.**

In the special case that u and v are *linear* expressions in x and y, it is possible to say more.

If $u = 0$ and $v = 0$ are the equations of two parallel straight lines, the equation

$$u + kv = 0, \qquad k \neq 0,$$

represents, in general, a straight line parallel to the given lines. For a single value of k, the equation has no locus.

Thus, if the parallel lines are

$$u \equiv x + y = 0, \qquad v \equiv x + y + 1 = 0,$$

the equation

(5) $$u + kv \equiv (1 + k)x + (1 + k)y + k = 0$$

* It may happen, also, that the equation represents just one point, as when, for example,

$$u = x^2 + y^2 - 2, \qquad v = x^2 + y^2 - 1, \qquad k = -2,$$

has no locus when $k = -1$, but otherwise it represents a line, of slope -1, parallel to the given lines. In fact, it yields all the lines of slope -1, except the line $v = 0$, since, if we rewrite it in the form,

$$x + y + \frac{k}{1+k} = 0, \qquad k \neq -1,$$

the quantity $k/(1+k)$ may be made to take on any value, except 1, by suitably choosing k.

Pencils of Curves. All the lines through a point, or all the parallel lines with a given slope, form what is called a *pencil of lines.* Equation (5) represents, when k is considered as an arbitrary constant, all the lines of slope -1, except the line

$$v \equiv x + y + 1 = 0;$$

in this case, then, $u + kv = 0$ and $v = 0$ together represent all the lines of slope -1, that is, *a pencil of parallel lines.*

Similarly, $u + kv = 0$, when $u = 0$ and $v = 0$ are the lines (1) and (2), yields all the lines through the point (1, 1), except the line (2); hence $u + kv = 0$ and $v = 0$ together represent all the lines through the point (1, 1), — *a pencil of intersecting lines.*

Thus, if $u = 0$ and $v = 0$ are any two lines, the equations

(6) $$u + kv = 0 \qquad \text{and} \qquad v = 0$$

together represent a pencil of lines.

If we set $k = m/l$ in $u + kv = 0$ and multiply by l, the resulting equation

(7) $$lu + mv = 0$$

is equivalent to the equation $u + kv = 0$ when $l \neq 0$, and when $l = 0$ $(m \neq 0)$, it becomes $v = 0$. Consequently, the two equations (6) may be replaced by the single equation (7).

The pencil of lines through the point (1, 1), for example, may now be given by the single equation

$$l(x + y - 2) + m(x - y) = 0,$$

where l and m have arbitrary values, not both zero.

In general, if $u=0$ and $v=0$ are any two curves, all the curves represented by the equation

$$lu+mv=0,$$

where l and m have arbitrary values, not both zero, form what is called *a pencil of curves.*

Applications. *Example* 1. Let

$$u\equiv x^2+y^2+ax+by+c=0,$$
$$v\equiv x^2+y^2+a'x+b'y+c'=0,$$

be the equations of any two circles which cut each other. Then the equation

$$u-v\equiv(a-a')x+(b-b')y+(c-c')=0$$

represents a curve which passes through the two points of intersection of the circles. But this equation, being linear, represents a straight line, and is, therefore, the equation of the *common chord* of the circles.

The foregoing proof is open to the criticism that conceivably we might have

$$a-a'=0, \qquad b-b'=0,$$

and then the equation $u-v=0$ would not represent a straight line. But in that case the circles would be concentric, and we have demanded that they cut each other.

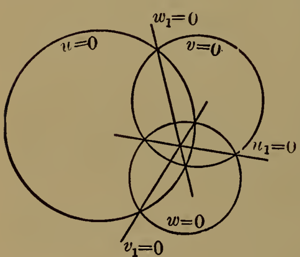

FIG. 4

Example 2. We can now prove the following theorem: *Given three circles, each pair of which intersect. Then their three common chords pass through a point, or are parallel.*

Let two of the three given circles be those of Example 1, and let the equation of the third circle be

$$w\equiv x^2+y^2+a''x+b''y+c''=0.$$

Then the equations of the three common chords can be written in the form:

$$u - v = 0, \qquad v - w = 0, \qquad w - u = 0.$$

Let

$$u_1 \equiv v - w, \qquad v_1 \equiv w - u, \qquad w_1 \equiv u - v.$$

We observe that the equation,

$$(8) \qquad u_1 + v_1 + w_1 \equiv 0 \qquad \text{or} \qquad -w_1 \equiv u_1 + v_1,$$

holds identically for all values of x and y. Consequently, the line $w_1 = 0$ is the same line as

$$u_1 + v_1 = 0,$$

and therefore it passes through the point of intersection of $u_1 = 0$ and $v_1 = 0$, or, if these lines are parallel, is parallel to them. Hence the theorem is proved.

The above proof is a striking example of a powerful method of Modern Geometry known as the *Method of Abridged Notation.** By means of this method many theorems, the proofs of which would otherwise be intricate, or for whose proof no method of attack is readily discerned, can be established with great ease.

EXERCISES

1. Find the equation of the straight line which passes through the origin and the point of intersection of the lines

$$2x - 3y - 2 = 0, \qquad 5x + 2y + 1 = 0.$$

Ans. $12x + y = 0$.

2. Find the equation of the straight line which passes through the point $(-1, 2)$ and meets the lines

$$x + y = 0, \qquad x + y + 3 = 0.$$

at their point of intersection.

* The first general development of this method was given by the geometer, Julius Plücker, in his *Analytisch-geometrische Entwicklungen* of 1828 and 1831.

3. Find the equation of the straight line which passes through the point of intersection of the lines

$$5x - 2y - 3 = 0, \qquad 4x + 7y - 11 = 0$$

and is parallel to the axis of y.

4. Find the equation of the straight line which passes through the point of intersection of the lines given in Ex. 3 and makes an angle of 45° with the axis of x.

5. Find the equation of the straight line which passes through the point of intersection of the lines of Ex. 1 and is perpendicular to the first of the lines given in Ex. 3.

Ans. $38x + 95y + 58 = 0$.

6. The same, if the line is to be parallel instead of perpendicular.

7. Find the equation of the common chord of the parabolas

$$y^2 - 2y + x = 0, \qquad y^2 + 2x - y = 0.$$

Ans. $x + y = 0$.

8. The same for the parabolas

$$2x^2 - 5x + 2y = 3,$$
$$3x^2 + 7x - 9y = 4.$$

9. Write the equation of the pencil of curves determined by the two curves (*a*) of Ex. 1; (*b*) of Ex. 3; (*c*) of Ex. 7.

10. What is the equation of the pencil of circles determined by the two circles

$$x^2 + y^2 - 2x - 1 = 0,$$
$$x^2 + y^2 + 4x - 1 = 0\,?$$

Draw a figure showing the pencil. Find the equation of that circle of the pencil which goes through the point (2, 4).

11. Find the equation of the pencil of parallel lines (*a*) of slope 1; (*b*) of slope -3; (*c*) of slope λ_0.

Ans. (*a*) $y = x + k$.

12. Find the equation of the pencil of lines through (*a*) the point (0, 0); (*b*) the point (3, 2); (*c*) the point $(0, b)$; (*d*) the point (x_0, y_0). *Ans.* (*a*) $lx + my = 0$.

4. The Equation $uv = 0$. Consider, for example, the equation

(1) $$x^2 - y^2 = 0.$$

Since $$x^2 - y^2 \equiv (x - y)(x + y),$$

it is clear that equation (1) will be satisfied

(*a*) if (x, y) lies on the line

(2) $$x - y = 0;$$

(*b*) if (x, y) lies on the line

(3) $$x + y = 0;$$

and in no other case. Equation (1), therefore, is equivalent to the two equations (2) and (3) taken together, and it represents, therefore, the two right lines (2) and (3).

It is clear from this example that we can generalize and say:

THEOREM. *The equation*

$$uv = 0$$

represents those points (x, y) *which lie on each of the two curves,*

$$u = 0, \qquad v = 0,$$

and no others.

It follows as an immediate consequence of the theorem that the equation

$$uvw \cdots = 0,$$

whose left-hand member is the product of any number of factors, represents the totality of curves corresponding to the individual factors, when these are successively set equal to zero.

Example. Consider the equation,

$$x^4 - y^4 = 0.$$

Here,*

$$x^4 - y^4 = (x^2 - y^2)(x^2 + y^2) = (x - y)(x + y)(x^2 + y^2).$$

* It is true that the following equation is an identity, and so the sign $\equiv$ instead of $=$ might be expected. The use of the sign $\equiv$ for an identical equation is not, however, considered obligatory, the sign $=$ being used when it is clear that the equation is an identity, so that the fact does not require special emphasis.

The given equation is, therefore, equivalent to the three equations:

$$x - y = 0, \qquad x + y = 0, \qquad x^2 + y^2 = 0.$$

The first two of these equations represent right lines. The third is satisfied by the coördinates of a single point, the origin. Since this point lies on the right lines, the third equation contributes nothing new to the locus.

EXERCISES

What are the loci of the following equations?

1. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$ **2.** $x^2 + 3x + 2 = 0.$

3. $2x^2 + 3xy - 2y^2 = 0.$ **4.** $xy + x + 2y + 2 = 0.$

5. $x^2 + xy - 2x - 2y = 0.$ **6.** $x^3 + xy^2 = x.$

7. $3x^2y - 2xy = 0.$ **8.** $x^4 - y^4 - 2x^2 + 2y^2 = 0.$

9. $(x + y - 1)(x^2 + y^2) = 0.$ *Ans.* The line whose intercepts on the axes are both 1, and the origin.

10. $(x + y)(x^2 + y^2 + 1) = 0.$

11. $(x + y)[(x - 1)^2 + y^2] = 0.$

12. $x^3 + x^2y - xy^2 - y^3 = 0.$

Find, in each of the following exercises, a single equation whose locus is the same as that of the given systems of equations.

13. $x - 2 = 0, \qquad y - 4 = 0.$

14. $x = 2, \qquad y = 4.$

15. $x + y - 2 = 0, \qquad x - y + 2 = 0.$

16. $x - 3y = 5, \qquad 4x + 3 = 0.$

17. $\frac{x}{a} = \frac{y}{b}, \qquad \frac{x}{a} = -\frac{y}{b}.$

5. Tangents with a Given Slope. Discriminant of a Quadratic Equation. From elementary algebra we know that the roots of the quadratic equation

(1) $$Ax^2 + Bx + C = 0, \qquad A \neq 0,$$

are

$$x_1 = -\frac{B}{2A} + \frac{1}{2A}\sqrt{B^2 - 4AC},$$

$$x_2 = -\frac{B}{2A} - \frac{1}{2A}\sqrt{B^2 - 4AC}.$$

From these formulas the truth of the following theorem at once becomes apparent.

THEOREM 1. *The roots of the quadratic equation* (1) *are equal if and only if*

$$B^2 - 4AC = 0.$$

The quantity $B^2 - 4AC$ is known as the *discriminant* of the quadratic equation (1).

By means of the theorem we shall solve the following problem.

Problem. Let it be required to find the equation of the tangent to the parabola

(2) $$y^2 = 6x,$$

which is of slope $\frac{1}{2}$.

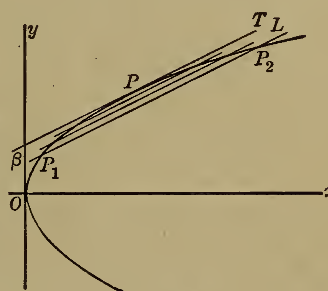

FIG. 5

Let L be a line of slope $\frac{1}{2}$ which meets the parabola in two points, P_1 and P_2. If we allow L to move parallel to itself toward the tangent, T, the points P_1 and P_2 will move along the curve toward P, the point of tangency of T; and if L approach T as its limit, the points P_1 and P_2 will approach the one point P as their limit.

It is clear that these considerations are valid for any conic. Accordingly, we may state the following theorem.

THEOREM 2. *A line which meets a conic intersects it in general in two points. If these two points approach coincidence* *in a single point, the limiting position of the line is a tangent to the conic.**

In applying Theorem 2 to the problem in hand, let us denote the intercept of the tangent T on the axis of y by β. The equation of T is, then,

(3) $$y = \tfrac{1}{2}x + \beta.$$

The coördinates of the point P, in which T is tangent to the parabola, are obtained by solving equations (2) and (3) simultaneously. Substituting in (2) the value of y given by (3), we have

$$(\tfrac{1}{2}x + \beta)^2 = 6x,$$

or

(4) $$x^2 + 4(\beta - 6)x + 4\beta^2 = 0.$$

The roots of equation (4) are equal, since they are both the abscissa of P. Accordingly, by Theorem 1, the discriminant of (4) is zero. Hence

$$16(\beta - 6)^2 - 16\beta^2 = 0, \qquad \text{or} \qquad -12\beta + 36 = 0.$$

Thus $\beta = 3$, and the tangent to the parabola (2) whose slope is $\frac{1}{2}$ has the equation

(5) $$x - 2y + 6 = 0.$$

If in (4) we set $\beta = 3$, the resulting equation,

$$x^2 - 12x + 36 = 0,$$

has equal roots, as it should. The common value is $x = 6$, and the corresponding value of y, from (2), is $y = 6$. The coördinates of the point of tangency, P, are, then, (6, 6).

Second Method. We proceed now to give a second method of solution for the type of problem just discussed. Let the conic be the ellipse

(6) $$4x^2 + y^2 = 5,$$

and let the given slope be 4.

* A tangent to a conic might then be defined as the limiting position of a line having two points of intersection with the conic, when these points approach coincidence in a single point; this is a generalization of

It is evident from the figure that there are two tangents of slope 4 to the ellipse. Let the intercept on the axis of y of one of the tangents be β. The equation of this tangent is then

(7) $$y = 4x + \beta.$$

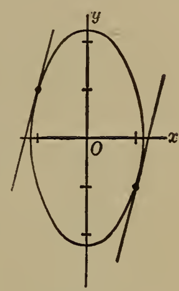

FIG. 6

Our problem now is to determine the value of β. To this end, let the coördinates of the point of contact of the tangent be (x_1, y_1). Then a second equation of the tangent is, by (12), § 2,

(8) $$4x_1x + y_1y = 5.$$

Since equations (7) and (8), which we rewrite as

$$4x - y + \beta = 0,$$
$$4x_1x + y_1y - 5 = 0,$$

represent the same line, it follows, from Ch. II, § 10, Th. 5, that

$$\frac{4x_1}{4} = \frac{y_1}{-1} = \frac{-5}{\beta}.$$

From the equality of the first and third ratios we have

(9) $$x_1 = -\frac{5}{\beta}.$$

Since the second and third ratios are equal,

(10) $$y_1 = \frac{5}{\beta}.$$

Furthermore, the point (x_1, y_1) lies on the ellipse and so the values of x_1 and y_1, given by (9) and (10), satisfy equation (6). Accordingly,

$$\frac{100}{\beta^2} + \frac{25}{\beta^2} = 5, \qquad \text{or} \qquad \frac{25}{\beta^2} = 1.$$

Hence β has the value 5 or -5.

the definition of § 1. A tangent cannot be defined as a line meeting the conic in a single point, for there are lines of this character which are not tangents, viz., a line parallel to the axis of a parabola, or to an asymptote of a hyperbola.

Substituting these values of β in turn in (7), we obtain

$$4x - y + 5 = 0, \qquad 4x - y - 5 = 0,$$

as the equations of the two tangents of slope 4 to the ellipse (6). From equations (9) and (10) it follows that the points of contact of these tangents are, respectively, $(-1, 1)$ and $(1, -1)$.

Both the methods described in this paragraph are general in application. For the usual type of problem met with in a first course in Analytic Geometry either method may be used with facility. It is, however, to be noted that the second method presupposes that the equation of the tangent to the curve at an arbitrary point on the curve is known, whereas the first does not. Accordingly, in case a curve is given, for which the general equation of the tangent is not known, — for example, the parabola, $y = 3x^2 - 2x + 1$, — the first method will be shorter to apply.

EXERCISES

Determine in each of the following cases how many tangents there are to the given conic with the given slope. Find the equations of the tangents and the coördinates of the points of tangency. Use both methods in Exs. 1, 2, 3, checking the results of one by those of the other.

	Conic	*Slope*
1.	$x^2 + y^2 = 5,$	2.
	Ans.	$\begin{cases} 2x - y - 5 = 0, \text{ tangent at } (2, -1), \\ 2x - y + 5 = 0, \text{ tangent at } (-2, 1). \end{cases}$
2.	$y^2 = 3x,$	$\frac{3}{2}$.
3.	$2x^2 + y^2 = 11,$	$-\frac{2}{3}$.
4.	$x^2 + 8y = 0,$	2.
5.	$4x^2 - y^2 = 20,$	3.
6.	$x^2 + y^2 + 2x = 0,$	$\frac{1}{2}$.
7.	$6y^2 - 5x = 0,$	$1\frac{2}{3}$.

8. What are the equations of the tangents to the circle

$$x^2 + y^2 = 10,$$

which are parallel to the line $3x - y + 5 = 0$?

9. What is the equation of the tangent to the ellipse

$$4x^2 + 5y^2 = 20,$$

which is perpendicular to the line $x + 3y - 3 = 0$ and has a positive intercept on the axis of y?

10. Find the equation of the tangent to the parabola

$$y = 3x^2 - 2x + 1,$$

which is perpendicular to the line $x + 4y + 3 = 0$.

Ans. $4x - y - 2 = 0$.

11. Make clear geometrically that, no matter what direction is chosen, there are always two tangents to a given ellipse, which have that direction.

12. How many tangents are there to the parabola $y^2 = 2mx$, which have the slope 0? State a general theorem relating to the number of tangents to a parabola which have a given slope.

13. Are there any tangents of slope 3 to the hyperbola

$$4x^2 - y^2 = 5?$$

If so, what are their equations?

14. The preceding exercise, if the given slope is (*a*) 1; (*b*) 2. Give reasons for your answers.

6. General Formulas for Tangents with a Given Slope. Consider first the hyperbola

$$(1) \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Before attempting to find a general formula for the equations of the tangents to the hyperbola, which have a given slope, λ, we shall do well to ask if such tangents exist. In answer to this question we state the following theorem.

THEOREM. *All the tangents to the hyperbola* (1) *are steeper than the asymptotes. Their slopes* λ *all satisfy the inequality*

$$(2) \qquad |\lambda| > \frac{b}{a} \qquad \text{or} \qquad \lambda^2 > \frac{b^2}{a^2}.$$

Conversely, if λ *satisfies* (2), *there are two tangents of slope* λ *to* (1). *If, however,* $\lambda^2 \leq b^2/a^2$, *there are no tangents of slope* λ *to* (1).

To prove the theorem, let a point P, starting from the vertex A, trace the upper half of the right-hand branch of (1). Then the tangent, T, at P, starting from the vertical position at A, turns continuously in one direction, and, as P recedes indefinitely, approaches the asymptote S as its limit. In other words, the slope, λ, of T decreases continuously through all positive values greater than the slope, b/a, of S, and approaches b/a as its limit.* Consequently, λ is always greater than b/a:

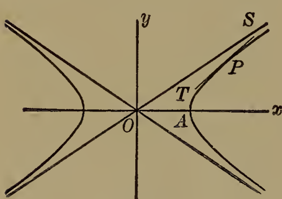

FIG. 7

$$\lambda > \frac{b}{a}.$$

* The geometrical evidence of this is convincing, but not conclusive. To clinch it, we give the following analytical proof: If the coördinates of P are (x, y), the slope λ of T is, by (11), § 2,

$$\lambda = \frac{b^2 x}{a^2 y}.$$

According to Ch. VIII, § 4, eq. (3), $\dfrac{x}{y} = \dfrac{a}{b} \dfrac{1}{\sqrt{1 - \dfrac{a^2}{x^2}}}$.

Hence
$$\lambda = \frac{b}{a} \frac{1}{\sqrt{1 - \dfrac{a^2}{x^2}}}.$$

When P traces the upper half of the right-hand branch of (1) and recedes indefinitely, x increases continuously from the value a through all values greater than a. Then a^2/x^2 decreases continuously from 1 and approaches 0 as its limit; and $1 - a^2/x^2$, and hence $\sqrt{1 - a^2/x^2}$, in-

If P now traces the lower half of the right-hand branch, λ is negative, and always:

$$-\lambda > \frac{b}{a}.$$

These two inequalities can be combined into the single inequality (2). Thus (2) is satisfied by the slope λ of every tangent to the right-hand branch of (1), and hence also, because of the symmetry of the curve, by the slope λ of every tangent to the left-hand branch.

From the reasoning given in the first case, when P traces the upper half of the right-hand branch of (1), it follows, not only that $\lambda > b/a$, but also that λ takes on *every* value greater than b/a. Hence, if a value of λ, greater than b/a, is arbitrarily chosen, there is surely at least one tangent of this slope λ to (1), and consequently, because of the symmetry of the curve, there are actually two. Similarly, if a value of λ less than $-b/a$ is given.

To find the equations of the two tangents of slope λ to (1), in the case that λ does satisfy (2), we apply the first of the two methods of § 5. Let the equation of one of the tangents be

(3) $$y = \lambda x + \beta,$$

where β is to be determined. Proceeding to solve (1) and (3) simultaneously, we substitute for y in (1) its value as given by (3) and obtain the equation,

$$b^2x^2 - a^2(\lambda x + \beta)^2 = a^2b^2,$$

or

(4) $$(b^2 - a^2\lambda^2)x^2 - 2a^2\beta\lambda x - a^2(b^2 + \beta^2) = 0.$$

The roots of equation (4) are both equal to the abscissa of the

creases continuously from 0 and approaches 1 as its limit. Consequently, the reciprocal, $1/\sqrt{1 - a^2/x^2}$, of $\sqrt{1 - a^2/x^2}$ decreases continuously through all positive values greater than 1 and approaches 1 as its limit. Hence, finally, λ decreases continuously through all positive values greater than b/a and approaches b/a as its limit, q. e. d.

point of contact of the tangent (3), and hence the discriminant of (4) must vanish. We have, then,

$$4a^4\beta^2\lambda^2 + 4a^2(b^2+\beta^2)(b^2 - a^2\lambda^2) = 0,$$

or, simplifying,

(5) $$\beta^2 = a^2\lambda^2 - b^2.$$

Hence β has either of the values

$$\pm\sqrt{a^2\lambda^2 - b^2},$$

and the equations of the two tangents, written together, are

(6) $$y = \lambda x \pm \sqrt{a^2\lambda^2 - b^2}.$$

Since λ satisfies (2), or the equivalent inequality $a^2\lambda^2 - b^2 > 0$, the quantity under the radical is positive and so has a square root.* We have thus obtained the following result.

The equations of the tangents to the hyperbola (1), *which have the given slope* λ, *where* λ *satisfies the inequality* (2), *are given by* (6).

Let the student deduce the following results, using either of the two methods of § 5.

The equations of the tangents to the ellipse

(7) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

which have an arbitrarily given slope λ, *are*

(8) $$y = \lambda x \pm \sqrt{a^2\lambda^2 + b^2}.$$

The equation of the tangent to the parabola

(9) $$y^2 = 2mx,$$

which has a given slope λ, *not* 0, *is*

(10) $$y = \lambda x + \frac{m}{2\lambda}.$$

* If we take a value of λ, for which $\lambda^2 < b^2/a^2$, then $a^2\lambda^2 - b^2$ is negative and has no square root. Consequently, there are no tangents with this slope, as the theorem states. Finally, if $\lambda = \pm b/a$, then $a^2\lambda^2 - b^2 = 0$, and (4) is not a quadratic equation.

Condition that a Line be Tangent to a Conic. The two methods used to find the tangent to a conic with a given slope apply equally well to the problem of determining the condition that an arbitrary line be tangent to a given conic. In fact, in finding the equations of the tangents of slope λ to the hyperbola (1), we have at the same time shown that *the condition that the line*

$$(11) \qquad y = \lambda x + \beta,$$

where we now consider λ and β both arbitrary, be tangent to the hyperbola (1), *is that λ and β satisfy the equation* (5):

$$\beta^2 = a^2\lambda^2 - b^2.$$

Similarly, the work of deriving formula (8) or (10) involves finding the condition that the line (11) be tangent to the ellipse (7) or the parabola (9).

Example. Is the line $3x - 2y + 5 = 0$ tangent to the hyperbola $x^2 - 4y^2 = 4$?

It is, if, when we write the equations of the line and the hyperbola in the forms (11) and (1), the values which we obtain for λ, β, a^2, and b^2, namely, $\frac{3}{2}$, $\frac{5}{2}$, 4, and 1, satisfy (5). It is seen that they do not, and hence the line is not tangent to the hyperbola.

EXERCISES

1. Derive formula (8) and at the same time show that the condition that the line (11), where now λ and β are both arbitrary, be tangent to the ellipse (7) is that λ and β satisfy the equation

$$(12) \qquad \beta^2 = a^2\lambda^2 + b^2.$$

2. Show that the line (11) is tangent to the parabola (9) if and only if

$$(13) \qquad 2\lambda\beta = m.$$

Hence prove the validity of formula (10).

3. By direct application of the methods of the text, show that the condition that the line (11) be tangent to the circle

(14) $$x^2 + y^2 = a^2$$

is that

(15) $$\beta^2 = a^2(1 + \lambda^2).$$

4. Using formulas (6), (8), and (10), find the equations of the tangents which are required in Exs. 1, 2, 3, 5, and 7 of § 5.

5. Has the hyperbola $9x^2 - 4y^2 = 36$ any tangents whose inclination to the axis of x is 60°? Whose inclination is 45°? If so, find their equations.

6. Find the equations of the tangents to the parabola $y^2 = 8x$, one of which is parallel to and the other perpendicular to the line $3x - 2y + 5 = 0$. Show that these tangents intersect on the directrix.

7. Prove that any two perpendicular tangents to a parabola intersect on the directrix.

In each of the following exercises determine whether the given line is tangent to the given conic. If it is, find the coördinates of the point of contact.

	Conic	*Line*
8.	$2x^2 + 3y^2 = 5,$	$2x - 3y - 5 = 0.$
9.	$y^2 = 2x,$	$x + 4y + 8 = 0.$
10.	$3x^2 - 5y^2 = 7,$	$6x - 5y - 8 = 0.$

In each of the following cases the equation of the given line contains an arbitrary constant. Find the value or values of this constant, if any exist, for which the line is tangent to the given conic.

	Conic	*Line*	
11.	$x^2 + 3y^2 = 4,$	$x - 3y + c = 0.$	*Ans.* $c = \pm 4.$
12.	$x^2 - y^2 = 3,$	$2x + dy - 3 = 0.$	
13.	$5y^2 = 3x,$	$kx - 10y + 15 = 0.$	
14.	$4x^2 - 3y^2 = 1,$	$x + 2y + k = 0.$	

15. Is the line $x + y = 1$ tangent to the parabola $y = x - x^2$?

16. Show that the lines $3x \pm y + 10 = 0$ are common tangents of the circle $x^2 + y^2 = 10$ and the parabola $y^2 = 120x$.

17. Find the equations of the common tangents of the parabola $y^2 = 4\sqrt{2}x$ and the ellipse $x^2 + 2y^2 = 4$.

Ans. $\sqrt{2}x \pm 2y + 4 = 0$.

7. Tangents to a Conic from an External Point. Given a point P external to a conic, that is, lying on the convex side of the curve. From P it is possible, in general, to draw two tangents to the conic. It is required to find the equations of these tangents.

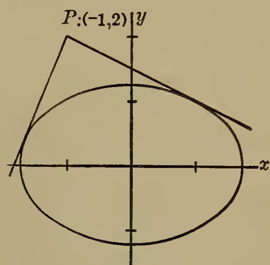

FIG. 8

Let the conic be the ellipse

(1) $$x^2 + 2y^2 = 3$$

and let P be the point $(-1, 2)$. We find the equations of the two tangents drawn from P to the ellipse by finding first the coördinates of the points of tangency. Let P_1 be the point of tangency of one of the tangents, and let the coördinates of P_1, which are as yet unknown, be (x_1, y_1). The equation of this tangent is then, by (12), § 2,

(2) $$x_1x + 2y_1y = 3.$$

There are two conditions on the point P_1, to serve as a means of determining the values of x_1 and y_1. In the first place, the tangent (2) at P_1 must go through the point $(-1, 2)$; hence

(3) $$-x_1 + 4y_1 = 3.$$

Secondly, the point P_1 lies on the ellipse (1); that is,

(4) $$x_1^2 + 2y_1^2 = 3.$$

Equations (3) and (4) are two simultaneous equations in the unknowns x_1, y_1. If we solve (3) for x_1:

(5) $$x_1 = 4y_1 - 3,$$

and substitute its value in (4), we obtain, on simplification, the following equation for y_1:

$$3y_1^2 - 4y_1 + 1 = 0. \tag{6}$$

The roots of this equation are $y_1 = 1$ and $y_1 = \frac{1}{3}$; the corresponding values of x_1 are, from (5), 1 and $-\frac{5}{3}$. Hence $(x_1, y_1) = (1, 1)$ and $(x_1, y_1) = (-\frac{5}{3}, \frac{1}{3})$ are the solutions of (3) and (4).

The coördinates of the points of tangency are, therefore, (1, 1) and $(-\frac{5}{3}, \frac{1}{3})$. Substituting the coördinates of each point in turn for $x_1\ y_1$ in (2) and simplifying the results, we obtain, as the equations of the two tangents,

$$x + 2y - 3 = 0 \quad \text{and} \quad 5x - 2y + 9 = 0. \tag{7}$$

The method used in this example is universal in its application, not only to conics, but to other curves as well. It should be noted, however, that the equation corresponding to (6) does not, in general, have rational, that is, fractional or integral, roots. Usually its roots involve radicals and hence so do the final equations of the tangents. If one were dealing with an arbitrary point P external to an arbitrary conic, for example, the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

these radicals would be complicated. Accordingly, we make no attempt to set up general formulas for the tangents to a given conic from an external point. We have expounded a method which is applicable in all cases, and this is the purpose we set out to achieve.

Second Method. We give briefly an alternative method of finding the equations of the tangents from the point $(-1, 2)$ to the ellipse (1).

Suppose one of the tangents is the line

$$y = \lambda x + \beta. \tag{8}$$

Since it is a tangent to (1), we have, according to § 6, Ex. 1,

$$2\beta^2 = 6\lambda^2 + 3.$$

Since it contains the point $(-1, 2)$,

$$2 = -\lambda + \beta.$$

If we solve these equations in λ and β simultaneously, we find that $\lambda = -\frac{1}{2}$ or $\frac{5}{2}$ and that $\beta = \frac{3}{2}$ or $\frac{9}{2}$. Substituting these pairs of values for λ and β in turn in (8) and simplifying the results, we obtain the equations (7).

EXERCISES

1. Make clear geometrically that from a point external to an ellipse or a parabola there can always be drawn just two tangents to the curve.

2. How many tangents can be drawn to a hyperbola from its center? From a point on an asymptote, not the center? From any other external point? Summarize your answers in the form of a theorem.

3. Let P be a point external to a hyperbola from which two tangents can be drawn to the curve. How must the position of P be restricted, if the two tangents are drawn to the same branch of the hyperbola? To different branches?

4. The point $(2, 0)$ is a point internal to the hyperbola $x^2 - 2y^2 = 2$. Prove analytically that no tangent can be drawn from it to the curve.

In each of the following exercises determine how many tangents there are from the point to the conic, and when there are tangents, find their equations. Use the first method.

	Conic	*Point*	
5.	$x^2 + y^2 = 5$,	$(3, 1)$.	*Ans.* $\begin{cases} x + 2y - 5 = 0, \\ 2x - y - 5 = 0. \end{cases}$
6.	$x^2 - 3y^2 = 4$,	$(\frac{1}{2}, -1)$.	
7.	$x^2 - 2y^2 = 2$,	$(1, -2)$.	
8.	$4x^2 - 9y^2 = 36$,	$(4, 1)$.	
9.	$y^2 - 4x = 0$,	$(4, 5)$.	
10.	$x^2 - 4y^2 = 4$,	$(2, 1)$.	

11. $x^2 - 8y = 0$, (3, 2).

12. $2x^2 - 3y^2 = -10$, (−2, 1).

13. $x^2 + y^2 - 4x - y = 0$, (5, 2).

14. $x^2 + y^2 = 25$, (−1, 7).

15. Work Exercises 5–10 by the second method.

16. Show, by use of the second method, that the tangents from the point (2, 3) to the ellipse $4x^2 + 9y^2 = 36$ are perpendicular.

EXERCISES ON CHAPTER IX

1. Prove that the slope of the conic

$$(1 - e^2)x^2 + y^2 - 2mx + m^2 = 0$$

at the point (x_1, y_1) is

$$\lambda = -\frac{(1 - e^2)x_1 - m}{y_1}.$$

Hence show that the equation of the tangent at (x_1, y_1) is

$$(1 - e^2)x_1x + y_1y - m(x + x_1) + m^2 = 0.$$

2. Show that the slope of the curve

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

at the point (x_1, y_1) is

$$\lambda = -\frac{2Ax_1 + By_1 + D}{Bx_1 + 2Cy_1 + E}.$$

Then prove that the equation of the tangent at (x_1, y_1) is

$$Ax_1x + \frac{B}{2}(y_1x + x_1y) + Cy_1y + \frac{D}{2}(x + x_1) + \frac{E}{2}(y + y_1) + F = 0.$$

3. The following equations contain arbitrary constants. What does each represent?

(a) $y = \lambda x + 3$;

Ans. All the lines through (0, 3) except $x = 0$.

(b) $3ax + 2y + a - 3 = 0$;

(c) $7x + 5y - c + 3 = 0$;

(d) $(2a + 5)x + (7a - 3)y = 9a + 2$;

(e) $lx+(2l+m)y-3m=0$;
(f) $(2l+3m)x-(4l+6m)y+5l=0$.

4. A line moves so that the sum of the reciprocals of its intercepts is constant. Show that it always passes through a fixed point.

5 A line with positive intercepts moves so that the excess of the intercept on the axis of x over the intercept on the axis of y is equal to the area of the triangle which the line forms with the axes. Show that it always passes through a fixed point.

6. Prove that the straight lines,

$$5x-2y+6=0,$$
$$2x-4y+3=0,$$
$$3x+2y+3=0,$$

meet in a point, by showing that the equation of one of them can be written in the form $lu+mv=0$, where $u=0$ and $v=0$ are the equations of the others.

7. Show that the three lines,

$$x+3y-4=0,$$
$$5x-3y+6=0,$$
$$3x-9y+14=0,$$

meet in a point.

8. Prove that the three lines

$$k\alpha-l\beta=0, \qquad l\beta-m\gamma=0, \qquad m\gamma-k\alpha=0,$$

where $\alpha=0$, $\beta=0$, and $\gamma=0$ are themselves equations of straight lines and k, l, and m are constants, meet in a point.

9. Find the equation of the common chord of the two intersecting circles

$$x^2+y^2+6x-8y+3=0,$$
$$2x^2+2y^2-3x+4y-12=0.$$

10. Show that the two circles,

$$x^2+y^2-4x-4y-10=0,$$
$$x^2+y^2+6x+6y+10=0,$$

are tangent to one another. Find the equation of the common tangent and the coördinates of the common point.

11. Find the equation of the circle which goes through the points of intersection of the two circles of Ex. 9 and through the origin.

12. Find the equation of the circle which is tangent to the circles of Ex. 10 at their common point and meets the axis of x in the point $x = 2$.

13. What is the equation of the circle which passes through the points of intersection of the line

$$2x - y + 4 = 0$$

and the circle

$$x^2 + y^2 + 2x - 4y + 1 = 0,$$

and goes through the point (1, 1)?

14. Determine the equation of the ellipse which passes through the points of intersection of the ellipse

$$x^2 + 4y^2 = 4$$

and the line

$$3x - 4y - 3 = 0,$$

and goes through the point (2, 1). By a transformation to parallel axes (cf. Ch. XI, § 1), prove that this ellipse has axes parallel to those of the given ellipse and has the same eccentricity.

15. Find a single equation representing both diagonals of the rectangle whose center is at the origin and one of whose vertices is at the point (a, b).

16. What is the condition that the equation

$$a^2x^2 - b^2y^2 = 0$$

represent two perpendicular lines?

17. Find the locus of each of the following equations:

(a) $6x^2 + 5xy - 4y^2 = 0;$

(b) $4x^2 - 20xy + 25y^2 = 0;$

(c) $x^2 + xy + 6y^2 = 0;$

18. Prove that the equation

$$(1) \qquad Ax^2 + Bxy + Cy^2 = 0$$

represents the origin, a single straight line, or two straight lines, according as the discriminant, $B^2 - 4AC$, is negative, zero, or positive.

19. Show that, if equation (1), Ex. 18, represents two straight lines, the slopes of these lines are the roots of the equation

$$C\lambda^2 + B\lambda + A = 0.$$

20. Prove that the equation

$$14x^2 - 45xy - 14y^2 = 0$$

represents two perpendicular straight lines.

21. Show that equation (1), Ex. 18, represents two perpendicular straight lines if and only if $A + C = 0$.

22. Prove that the equation

$$y^2 - 2xy \sec\theta + x^2 = 0$$

represents two straight lines which form with one another the angle θ.

23. A regular hexagon has its center at the origin and two vertices on the axis of x. Find a single equation which represents all three diagonals. *Ans.* $y^3 - 3x^2y = 0$.

24. Determine the points of contact of the tangents drawn to an ellipse from the points on the conjugate axis which are at a distance from the center equal to the semi-axis major.

25. Find the equations of the common tangents of each of the following pairs of conics:

(*a*) $x^2 + y^2 = 16, \qquad y^2 = 6x;$

(*b*) $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1, \qquad \dfrac{x^2}{16} + \dfrac{y^2}{25} = 1;$

(*c*) $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1, \qquad \dfrac{x^2}{25} - \dfrac{y^2}{16} = 1.$

Draw a good figure in each case, showing the common tangents.

26. Show that the line

$$\frac{x}{A}+\frac{y}{B}=1 \tag{2}$$

is tangent to the circle

$$x^2+y^2=a^2$$

if and only if $$\frac{1}{A^2}+\frac{1}{B^2}=\frac{1}{a^2}.$$

27. Find the condition that the line (2), Ex. 26, be tangent to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad \textit{Ans. } \frac{a^2}{A^2}+\frac{b^2}{B^2}=1.$$

28. What will the condition obtained in Ex. 27 become in the case of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\,?$$

29. Prove that the line (2), Ex. 26, is tangent to the parabola $y^2=2\,mx$, if and only if $2\,B^2+Am=0$.

30. Find the condition that the line $y=\lambda x+\beta$ be tangent to the conic

$$(1-e^2)x^2+y^2-2\,mx+m^2=0.$$

Ans. $(\beta+m\lambda)^2-e^2(\beta^2+m^2)=0.$

31. In an ellipse there is inscribed a rectangle with sides parallel to the axes. In this rectangle there is inscribed a second ellipse, with axes along the axes of the first. Show that a line joining extremities of the major and minor axes of the first ellipse is tangent to the second.